

Homework 6 for MATH5070

Topology of Manifolds

Due Wednesday, Nov. 27

- (Mayer-Vietoris) In Figure 1 below, all squares commute. The rows are exact, and in the columns the image of every arrow is in the kernel of the next arrow.

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & C_1^{k+1} & \xrightarrow{i} & C_2^{k+1} & \xrightarrow{j} & C_3^{k+1} \longrightarrow 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 0 & \longrightarrow & C_1^k & \longrightarrow & C_2^k & \longrightarrow & C_3^k \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & C_1^{k-1} & \xrightarrow{i} & C_2^{k-1} & \xrightarrow{j} & C_3^{k-1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow
 \end{array}$$

Show that the Mayer-Vietoris sequence associated with Figure 1

$$\longrightarrow H^k(C_1) \xrightarrow{i_*} H^k(C_2) \xrightarrow{j_*} H^k(C_3) \xrightarrow{\delta} H^{k+1}(C_1) \longrightarrow$$

is exact.

- (The Five Lemma) In Figure 2 below, all the arrows commute. The rows are exact and the vertical arrows α , β , δ and ϵ are isomorphisms. Show that the middle arrow, γ , is an isomorphism

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \delta \uparrow & & \epsilon \uparrow \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

3. (Poincaré Duality) Let M be an oriented n -dimensional manifold with finite topology, i.e., M admits a finite good cover. From the pairing

$$H_c^{n-k}(M) \times H^k(M) \rightarrow H_c^n(M) \rightarrow \mathbb{R}$$

one gets a map

$$H^k(M) \rightarrow H_c^{n-k}(M)^*.$$

Prove that this map is bijective.

Hint: Induction. Assume this is true if M admits a good cover by $N - 1$ open sets, and prove that it is true if M admits a good cover by N open sets. To go from $N - 1$ to N , use exercises 1 and 2.

4. (Optional) (Čech versus De Rham) Prove De Rham's theorem: If M is a connected n -dimensional manifold with finite topology and $\mathbb{U} = \{U_1, \dots, U_N\}$ is a good cover, then

$$H_{DR}^k(M) \cong \check{H}^k(\mathbb{U}, \mathbb{R}).$$

Hint: Some strenuous diagram-chasing using the **Weyl diagram** below.

Here, $C^{k,l} = \check{C}^k(\mathbb{U}, \Omega^l)$ and $\check{C}^k = \check{C}^k(\mathbb{U}, \mathbb{R})$. The vertical arrows are d 's and the horizontal arrows are δ 's. All columns are exact except the left-hand column which is the usual De Rham complex, and all rows are exact except the bottom one which is the usual Čech complex. All arrows commute.

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \Omega^2(M) & \longrightarrow & C^{0,2} & \longrightarrow & C^{1,2} & \longrightarrow & C^{2,2} & \longrightarrow & C^{3,2} & \longrightarrow & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & \uparrow d & & & \\
 0 & \longrightarrow & \Omega^1(M) & \longrightarrow & C^{0,1} & \longrightarrow & C^{1,1} & \longrightarrow & C^{2,1} & \longrightarrow & C^{3,1} & \longrightarrow & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
 0 & \longrightarrow & \Omega^0(M) & \longrightarrow & C^{0,0} & \longrightarrow & C^{1,0} & \longrightarrow & C^{2,0} & \longrightarrow & C^{3,0} & \longrightarrow & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
 & & 0 & \longrightarrow & \check{C}^0 & \longrightarrow & \check{C}^1 & \longrightarrow & \check{C}^2 & \longrightarrow & \check{C}^3 & \longrightarrow & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & &
 \end{array}$$

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