MATH5021 THEORY OF PDE I
Homework 1

Due Oct 3, 2024, in hard copies. Late assignment is not accepted.

1. (5 points) Let $n > 1$. Show that the function
$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$
belongs to $W^{1,n}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is the unit disk centered at the origin. Use this to conclude that $W^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$.

2. (5 points) Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ that is bounded in some directions. Let $u \in H^1(\Omega)$ satisfying $u|_{\partial\Omega} = 0$. Prove that
$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},$$
where C is a positive constant. This is the Poincaré's inequality.
Hint: Without loss of generality, you may assume that Ω is bounded in the x_1-direction, say $0 < x_1 < a$ for some $a > 0$. Then
$$u(x_1, \cdot) = \int_0^{x_1} \partial_1 u(s, \cdot)\, ds.$$

3. [10 points] Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ satisfy the Maxwell's equations
$$\begin{cases} \partial_t \mathbf{E} = \nabla \times \mathbf{B}, \quad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}, & \text{in } [0,T] \times \mathbb{R}^3, \\ \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0, & \text{in } [0,T] \times \mathbb{R}^3, \\ \mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{B}|_{t=0} = \mathbf{B}_0. \end{cases}$$
Do the following:

   I. [5 points] Prove that
   $$\partial_t^2 \mathbf{E} - \Delta \mathbf{E} = 0, \quad \text{and } \partial_t^2 \mathbf{B} - \Delta \mathbf{B} = 0.$$
   Hint: Let X be a vector field. Then it holds that $\nabla \times (\nabla \times X) = \nabla(\nabla \cdot X) - \Delta X$.

   II. [5 points] Assume the initial data $\mathbf{E}_0, \mathbf{B}_0 \in H^1(\mathbb{R}^3)$. Prove that there exists $T > 0$ such that
   $$\|\nabla_{t,x}\mathbf{E}(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla_{t,x}\mathbf{B}(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} \leq \mathcal{C},$$
   holds for all $t \in [0,T]$, where $\mathcal{C}$ depends on $\|\mathbf{E}_0\|_{H^1(\mathbb{R}^3)}$ and $\|\mathbf{B}_0\|_{H^1(\mathbb{R}^3)}$.

4. (5 points) Consider the 3D Laplace operator $\Delta = \sum_{i=1}^3 {\partial_i}^2$. Find its invariant representation under the spherical coordinates.