Real Analysis 24-10-18
Chap 3 Lebesgue measures and Hausdorff measures.
§ 3.1 Lebesgue measures on
$$\mathbb{R}^n$$
.
Let $R = \pi_i [a_i, b_i]$ be a closed rectangles in \mathbb{R}^n .
Let $|R|$ denote the volume of R , i.e. $|R| = \pi_i (b_i - a_i)$.
A cube in \mathbb{R}^n is a rectangle R such that $b_j - a_j$ are the same.
Def. The n-dim Lebesgue measure on \mathbb{R}^n is defined by
 $\int_{-\infty}^{\infty} (E) = -i h f \left\{ \sum_{i=1}^{\infty} |R_i| : |Q_i| > E, R_i closed cubes \right\}$
. $\int_{-\infty}^{n}$ is generated by the gauge (R, H) , where
 R is the collection of closed cubes in \mathbb{R}^n , $|R|$ is the volume
of R .

Alternatively, Lebesque measure can be constructed using
Riesa representation Than.
To see this, we define an operator L on
$$C_c(\mathbb{R}^n)$$
 by
 $L(f) = \int f dx$, $f \in C_c(\mathbb{R}^n)$
 \uparrow
(Riemann integral via Darboux Sum)
 $\xrightarrow{x_i} \mathbb{R}^n$
 $\sum_i f(x_i) \cdot (Ax_i)$
• $L(f)$ is well-defined (any $f \in C_c(\mathbb{R}^n)$ is
Riemann integrable)
• L is linear
• L is possitive Linear functional over
 $C_c(\mathbb{R}^n)$.

By Riesz representation Thm,
$$\exists a \text{ Borel measure } \mu_R$$

on R^n such that μ_R is finite on Compact sets
and
 $L(f) = \int f d \mu_R$
Leb. integration.

Prop 3.1 $\mu_R = d^n$.

Pf. We first prove that μ_R is translation in Vaniant.
Let G be open in R^n , let $f \in C_c(R^n)$ such that
 $f < G$.
Let $x_0 \in R^n$, define $g(x) = f(x + x_0)$. It is direct
to see that
 $g < G - x_0$.

Moreover (using the def of R.I.)
 $\int_G f dx = \int_G g dx$
 $G - x_0$

Hence
$$L(f) = L(g)$$
, so

$$\mu_{R}(G) = \mu_{R}(G - x_{o}). \qquad (using \quad \mu_{R}(G) = sup \{ L(f) \\ = f < G \} \}$$
Using the outer regularity of μ_{R} , we see that

$$\mu_{R}(E) = \mu_{R}(E - x_{o}) \quad \text{for any } E \subset IR^{h}.$$
Hence μ_{R} is translation invariant.
Therefore

$$\mu_{R} = C \cdot d^{h} \quad \text{for some } C \ge 0.$$
Next we prove $C=I$.
Let $R_{o} = (0, 1)^{h}$, let $R_{E} = (2, 1 - 2)^{h}$
for $0 < 2 < \frac{1}{2}$
For $\overline{R_{2}} < f < R_{0}$,

$$\mu_{R}(R_{0}) \ge L(f) = \int f dx \ge I \overline{R_{E}} I = (I - 2E)^{h}$$
 $(become \ f \equiv I \ on \ R_{E}$
and $f \ge 0$)
Letting $\Sigma \Rightarrow 0$ gives $\mu_{R}(R_{0}) \ge 1$.

However for any
$$f \in R_0$$
,

$$\int f dx = \int_{R_0} f dx \leq |R_0| \quad (f \text{ is supported} on R_0 \\ a \neq f \leq 1)$$
Hence

$$\mu_R(R_0) = \sup \{ L(f) : f < R_0 \} \leq |R_0| = 1$$
Therefore $\mu_R(R_0) = 1 = d^n(R_0) \Rightarrow C = 1$.

$$M_R(R_0) = 1 = d^n(R_0) \Rightarrow C = 1.$$
Prop. 3.2 Let $T : |R^n \Rightarrow R^n$ be a linear transformation
Then $\exists a \text{ constant } \Delta(T) \in |R|$ such that

$$d^n(TE) = \Delta(T) d^n(E), \quad \forall E \in R^n.$$
Furthermore $\Delta(T) = 1$ if T is a rotation or
a reflection.
Pf. Define

$$\mu(E) = d^n(TE), \quad E \in R^n.$$
Clearly μ is an outer measure on $|R^n$.

Next we assume T is non-singular. Notice that
for x, y \in R^h,

$$x - y = T^{-1}(Tx - Ty)$$
Hence

$$\|x - y\| \leq \|T^{-1}\| \cdot \|Tx - Ty\|$$

$$(\|\|a\|\| = \sqrt{a_1^2 + \dots + a_n^2}$$
for $\overline{a} = (a_1, \dots, a_n)$)
It implies $\|Tx - Ty\| \geq \frac{1}{\|T^{-1}\|} \cdot \|x - y\|$
Hence for A, B $\subset \mathbb{R}^n$
 $d(A, B) > 0 \Rightarrow d(TA, TB) > 0$
It implies that μ is a metric outer measure
 $(Recd) \ \mu(E) = d^n(TE))$
Therefore $\mu = C \cdot d^n$ for some constant.
Next assume T is singular, then $T(\mathbb{R}^n)$ is
contained in a hyplane of \mathbb{R}^n so,

$$L^{n}(r(\mathbb{R}^{n})) = 0. \quad \text{Then } \mu = 0$$

In the cone that T is a rotation or a reflection,
then

$$B = T(B) \quad \text{where } B \text{ is the unit ball centered}$$
at the origin.
So $\mu(B) = d^{n}(TB) = d^{n}(B) \implies C = 1.$
[4]
§ 3.1 Lebesque measure on IR.
Non-measurable subsets of IR.
(a: Is every subset of IR measurable wrt

$$d^{2}?$$
No. Answered by Vitali at 1905.

Vitali's construction of non-measurable (ots.
We introduce a relation "~" on IR by

$$x \sim y$$
 if and only $x-y \in \mathbb{Q}$
(the set of rationals)
One can check that \sim is an equivalence relation
 $\cdot x \sim y \Rightarrow y \sim x$
 $\cdot x \sim y, y \sim z \Rightarrow x \sim z$
 $\cdot x \sim x$
Then
 $IR = \coprod E_d$ where E_d are equivalence
 $classes$.
By the axiom of choice, by picking x_d from E_d
for each d ,
we form a set $\{x_a\}_d =: E$
We call E a Vitali set. Then as a remarkable

fact, we have

$$R = \bigcup_{q \in Q} (\xi + q)$$
where the union is disjoint. (check it).
Prop 3: Every set in R with positive Lebesgue measure
Contains a non-measurable subset.
Pf. It is equivalent to prove the followig:
If ACR such that all subsets of A are
measurable, then $d^{1}(A) = 0$.
Fix such a set A. For $q \in Q$, we define
 $Aq = A \cap (\xi + q)$,
where ξ is a Vitali set. Then
 $A = \bigsqcup_{q \in Q} Aq$
 $g \in Q$
 $(because R = \bigsqcup_{q \in Q} (\xi + q))$

Since
$$A_{g}$$
 are measurable by assumption, we have

$$J_{v}^{1}(A) = \sum_{q \in Q} J_{v}^{1}(A_{q})$$
Now we need to show that $J_{v}^{1}(A_{q}) = 0$ for each
 $g \in Q$.
By the inner regularity of J_{v}^{1} , it is enough to
show that
 $J_{v}^{1}(K) = 0$ for each compact subset
 $K = fA_{q}$.
Fix such a compact $K \subset A_{q}$. Define
 $H = \bigcup_{\substack{i \in Q \\ |ii| \leq 1}} K + r$
For rationals $r_{i} \neq r_{v}$, Since $K + r_{i} \subset E + q + r_{i}$. (they are
 $K + r_{v} \subset E + q + r_{v}$.
Hence the union in defining H is disjoint
So $J_{v}^{1}(H) = \sum_{\substack{i \in Q \\ |ii| \leq 1}} J_{v}^{1}(K + r) = \sum_{\substack{i \in Q \\ |ii| \leq 1}} J_{v}^{1}(K)$.

If
$$d^{4}(K) > 0 \Rightarrow d^{4}(H) = +\infty$$

however this is impossible because H is a bounded set
Hence $d^{4}(K) = 0$. [2].
Remark: Any Vitali set Σ is non-measurable.
Pf. Suppose on the contrary Σ is measurable.
 $R = \bigsqcup_{\substack{i \in \mathbb{Q} \\ i \in \mathbb{Q}}} (\Sigma + 9)$
 $(disjoint)$ measurable
Hence $d^{4}(R) = \sum_{\substack{i \in \mathbb{Q} \\ i \in \mathbb{Q}}} d(\Sigma + 9)$
 $= \sum_{\substack{i \in \mathbb{Q} \\ i \in \mathbb{Q}}} d(\Sigma)$
which implies $d(\Sigma) > 0$.

However for any compack subset K of
$$\mathcal{E}$$
,
Using the same trick,
 $H = \bigsqcup_{\substack{k \neq r \\ r \in \mathbb{Q} \\ (r| \leq 1)}} K + r$
 H is bild, $\Rightarrow d^{1}(K) = 0$
By the inner regularity of $\mathcal{E} \Rightarrow d^{1}(\mathcal{E}) = 0$.
• Cantor set and Canter function.
• Cantor set and Canter function.
• Middle-third Cantor set C. The construction is
the followig: $0 \vdash \frac{1}{2}$ Co
 $C_{0} \Rightarrow C_{1} \Rightarrow C_{2} \cdots \Rightarrow C_{n} \Rightarrow \cdots$
• $f \Rightarrow f \Rightarrow r r C_{2}$
Let $C = \bigcap_{n=0}^{\infty} C_{n}$

We call C the middle-third Cantor set.

Below we list some properties of C

- · C is compact, non-empty, un countable
- · nowhere dense
- · perfect (no isolated points)

$$\cdot \quad \mathcal{L}^{1}(\mathcal{C}) = 0$$

$$\cdot \quad \mathcal{C} = \left\{ \begin{array}{c} x = \sum_{n=1}^{\infty} Q_{n} \overline{\beta}^{n} : Q_{n} = 0 \text{ or } 2 \end{array} \right\}$$

Define
$$F: C \rightarrow [o, 1]$$
 by
 $\chi = \sum_{n=1}^{\infty} a_n \overline{3}^n \longmapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot 2^n$

. Then the function
$$F$$
 is continuous,
increasing, surjective.
. Moreover if (a,b) is an interval in
the complement of C_n , then
 $F(a) = F(b)$
For instance $a = \frac{1}{3}, b = \frac{a}{3}$
 $0 + \frac{a}{3} + \frac{a$

Now we extend
$$F: C \rightarrow [o, 1]$$

to $F: [0, 1] \rightarrow [o, 1]$
Such that F is constant in each component
of $[0, 1] \setminus C$
We call $F: [o, 1] \rightarrow [0, 1]$ the Cantor function
Then F satisfies the following properties:
D F is increasing on $[o, 1]$
P F is cts
F(x) = 0 for d^{1} -are $x \in [0, 1]$
F(C) = $[o, 1]$

Next we build an invertible cts function
$$h: [0, 1]$$

 $\rightarrow [0, 2]$
such that h^{-1} maps some non-measurable sets
into measurable subset.
Let $h(x) = x + F(x)$ for $x \in [0, 1]$
Notice that h is cts and strictly increasing.
 $h(0) = 0 + F(0) = 0$, $h(1) = 1 + F(1) = 2$.
So h is a homeomorphism from $[0, 1]$ to $[0, 2]$.
Key point : $d^{1}(h(C)) = 1$.
Notice that $C = \bigcap_{n=1}^{\infty} C_{n}$, $C_{n} V$
So $h(C) = \bigcap_{n=1}^{\infty} h(C_{n})$, $h(C_{n}) V$
By the continuity of deb. measure
 $d^{1}(h(C)) = \lim_{n \to \infty} d^{1}(h(C_{n}))$.

Notice that for each basic interval in Cn 3^{-h} (there are 2" basic Intervals in Cn) $\mathcal{J}^{1}(h(a,b)) = 2^{n} + 3^{n}$ $\left(b-\alpha=\overline{\beta}^{n}\right)$ $f_{x}(x) = x + F(x)$ $\mathcal{L}^{4}(\mathfrak{K}(\alpha,b)) = b + F(b) - \alpha - F(\alpha)$ $= (b-\alpha) + (F(b) - F(a))$ $= 3^{-h} + 2^{-h} (using the 3-adic expansions$ of b, a.)I leave it as an exer. Hence $d^{1}(h(C_{n})) = 2^{n}(z^{-n}+3^{-n})$ $= \left(+\left(\frac{2}{3}\right)^{n}\right)$ Lettý n→∞ gives d(h(c))=1..

Now by Prop 3.3,
$$\exists$$
 a non-measurable
 $A = h(C)$
Let $B = h^{-1}(A) = C$.
So $h(B) = A$, with A being non-measurable.
However $B = C$, So $d^{1}(B) = 0 \implies B$ is
measurable.
Remark : If $f : [a,b] \rightarrow [c,d]$
is a homeomorphism
then f maps every Borel subset
of $[a,b]$ into a Borel subset.
of $[c,d]$.
(Exer.)