Exercise 6

Many problems are taken from [R].

- In the proof of Lusin's (Theorem 2.12) it was assumed that f is non-negative, bounded and A is compact. Complete the proof by showing the conclusion still holds when f is finite a.e. and A is of finite measure.
- (2) Let μ be a Riesz measure on \mathbb{R}^n . Show that for every measurable function f, there exists a sequence of continuous function $\{f_n\}$ such that $f_n \to f$ almost everywhere.
- (3) Here we construct a Cantor-like set, or a Cantor set with positive measure, by modifying the construction of the Cantor set as follows. Let $\{a_k\}$ be a sequence of positive numbers satisfying

$$\gamma \equiv \sum_{k=1}^{\infty} 2^{k-1} a_k < 1.$$

Construct the set S so that at the *k*th stage of the construction one removes 2^{k-1} centrally situated open intervals each of length a_k . Establish the facts:

- (a) $\mathcal{L}^1(\mathcal{S}) = 1 \gamma$,
- (b) \mathcal{S} is compact and nowhere dense,
- (c) \mathcal{S} is perfect and hence uncountable.

Note. A set A is perfect if for every $x \in A$ and $\varepsilon > 0$, $(B_{\varepsilon}(x) \setminus \{x\}) \cap A \neq \phi$, that is, every point in A is an accumulation point of A. It is known that a perfect set must be uncountable.

- (4) Let $0 < \varepsilon < 1$. Construct an open set $G \subset [0,1]$ which is dense in [0,1] but $\mathcal{L}^1(G) = \varepsilon$.
- (5) Let A be the subset of [0,1] which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}^1(A)$.
- (6) Let \mathcal{N} be a Vitali set in [0,1]. Show that $\mathcal{M} = [0,1] \setminus \mathcal{N}$ has measure 1 and hence deduce that

$$\mathcal{L}^{1}(\mathcal{N}) + \mathcal{L}^{1}(\mathcal{M}) > \mathcal{L}^{1}(\mathcal{N} \cup \mathcal{M}).$$

(7) Let E be a subset of \mathbb{R} with positive Lebsegue measure. Prove that for each $\alpha \in (0, 1)$, there exists an open interval I so that $\mathcal{L}^1(E \cap I) \ge \alpha \mathcal{L}^1(I)$. It shows that E contains almost a whole interval. Hint: Choose an open G containing E such that $\mathcal{L}^1(E) \geq \alpha \mathcal{L}^1(G)$ and note that G can be decomposed into disjoint union of open intervals. One of these intervals satisfies our requirement.

- (8) Let *E* be a measurable set in \mathbb{R} with respect to \mathcal{L}^1 and $\mathcal{L}^1(E) > 0$. Show that E E contains an interval (-a, a), a > 0. Hint:
 - (a) U, V open, with finite measure, $x \mapsto \mathcal{L}^1((x+U) \cap V)$ is continuous on \mathbb{R} .
 - (b) A, B measurable, $\mu(A), \mu(B) < \infty$, then $x \mapsto \mathcal{L}^1((x+A) \cap B)$ is continuous. For $A \subset U, B \subset V$, try

$$|\mathcal{L}^1((x+U)\cap V) - \mathcal{L}^1((x+A)\cap B)| \le \mathcal{L}^1(U\setminus A) + \mathcal{L}^1(V\subset B).$$

- (c) Finally, $x \mapsto \mathcal{L}^1((x+E) \cap E)$ is positive at 0 and if $(x+E) \cap E \neq \phi$, then $x \in E \setminus E$.
- (9) Give an example of a continuous map φ and a measurable f such that f φ is not measurable. Hint: The function h = x + g(x) where g is the Cantor function is a continuous map from [0, 1] to [0, 2] with a continuous inverse.