

# Fall 2024 MATH5011 Real Analysis I

## Exercise 2

Notations in lecture notes are in use.

- (1) Let  $g$  be a measurable function in  $[0, \infty]$ . Show that

$$m(E) = \int_E g d\mu$$

defines a measure on  $\mathcal{M}$ . Moreover,

$$\int_X f dm = \int_X fg d\mu, \quad \forall f \text{ measurable in } [0, \infty].$$

- (2) Let  $\{f_k\}$  be measurable in  $[0, \infty]$  and  $f_k \downarrow f$  a.e.,  $f$  measurable and  $\int f_1 d\mu < \infty$ . Show that

$$\lim_{k \rightarrow \infty} \int f_k d\mu = \int f d\mu.$$

What happens if  $\int f_1 d\mu = \infty$ ?

- (3) Let  $f$  be a measurable function. Show that there exists a sequence of simple functions  $\{s_j\}$ ,  $|s_1| \leq |s_2| \leq |s_3| \leq \dots$ , and  $s_k(x) \rightarrow f(x)$ ,  $\forall x \in X$ .
- (4) Let  $\mu(X) < \infty$  and  $f$  be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f d\mu \in [a, b], \quad \forall E \in \mathcal{M}, \mu(E) > 0$$

for some  $[a, b]$ . Show that  $f(x) \in [a, b]$  a.e.

- (5) Let  $f$  be Lebesgue integrable on  $[a, b]$  which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every  $c$ . Show that  $f$  is equal to 0 a.e..

(6) Let  $f \geq 0$  be integrable and  $\int f d\mu = c \in (0, \infty)$ . Prove that

$$\lim_{n \rightarrow \infty} \int n \log \left( 1 + \left( \frac{f}{n} \right)^\alpha \right) d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0, 1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

(7) Let  $\mu(X) < \infty$  and  $f_k \rightarrow f$  uniformly on  $X$  and each  $f_k$  is bounded. Prove that

$$\lim_{k \rightarrow \infty} \int f_k d\mu = \int f d\mu.$$

Can  $\mu(X) < \infty$  be removed?

(8) Give another proof of Borel-Cantelli lemma in Problem 7, Ex.1, by using integration theory. Hint: Study  $g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x)$ .

(9) Let  $f$  be a Riemann integrable function on  $[a, b]$  and extend it to  $\mathbb{R}$  by setting it zero outside  $[a, b]$ .

(a) Show that  $f$  is Lebesgue measurable.

(b) Show that the Riemann integral of  $f$  is equal to  $\int_{\mathbb{R}} f d\mathcal{L}^1$ .

(c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on  $[a, b]$  and converges pointwisely to some function which is not Riemann integrable.

(10) Let  $f$  be integrable in  $(X, \mathcal{M}, \mu)$ . Show that for each  $\varepsilon > 0$ , there is some  $\delta$  such that

$$\int_E |f| < \varepsilon, \quad \text{whenever } E \in \mathcal{M}, \mu(E) < \delta.$$

This is called the absolute continuity of an integrable function.