Exercise 10

(1) Let $\mathcal{L}^1$ be the Lebesgue measure on $(0,1)$ and μ the counting measure on $(0,1)$. Show that $\mathcal{L}^1 \ll \mu$ but there is no $h \in L^1(\mu)$ such that $d\mathcal{L}^1 = h\,d\mu$. Why?

(2) Let μ be a measure and λ a signed measure on $(X, \mathfrak{M})$. Show that $\lambda \ll \mu$ if and only if $\forall \varepsilon > 0$, there is some $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ whenever $|\mu(E)| < \delta$, $\forall E \in \mathfrak{M}$.

(3) Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Show that

$$\int f\,d\lambda = \int fh\,d\mu, \quad \forall f \in L^1(\lambda),\ fh \in L^1(\mu)$$

where $h = \dfrac{d\lambda}{d\mu} \in L^1(\mu)$.

(4) Let μ, λ and ν be finite measures, $\mu \gg \lambda \gg \nu$. Show that $\dfrac{d\nu}{d\mu} = \dfrac{d\nu}{d\lambda}\dfrac{d\lambda}{d\mu}$, μ a.e.

(5) Show that the completion of $C_c(X)$ under the sup-norm is $C_0(X)$ where X is a locally compact, Hausdorff space.

(6) Provide a proof of Proposition 5.8.

(7) Show that $M(X)$, the space of all signed measures defined on $(X, \mathfrak{M})$, forms a Banach space under the norm $\|\mu\| = |\mu|(X)$.

(8) Show that $M_r(X)$ is a closed subspace in $M(X)$ on $(X, \mathcal{B})$ where X is a locally compact Hausdorff space. Hence it is a Banach space.