Suggested Solution 6

(1) Show that $\mathcal{C} = \{x \in [0,1] : x = 0.a_1a_2a_3\cdots,\ a_j \in \{0,2\}$ in one of its ternary expansion.$\}$.

Solution: Write $\mathcal{C}_0 = [0,1]$, $\mathcal{C}_1 = [0,1/3] \cup [2/3,1]$, etc. Then clearly we have $\mathcal{C}_0 = \{x \in [0,1] : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,1,2\}$, in one of its ternary expansion.$\}$ and $\mathcal{C}_1 = \{x \in [0,1] : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_1 \in \{0,2\},\ a_k \in \{0,1,2\}\ \forall k > 1$, in one of its ternary expansion.$\}$ and so on. Inductively we have $\mathcal{C}_n = \{x \in [0,1] : x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,2\}\ \forall k \le n,\ a_k \in \{0,1,2\}\ \forall k > n$, in one of its ternary expansion.$\}$ and the result follows if we take the intersection over $n \in \mathbb{N}$.

(2) Let $0 < \varepsilon < 1$. Construct an open set $G \subset [0,1]$ which is dense in $[0,1]$ but $\mathcal{L}^1(G) = \varepsilon$.

Solution: Similar to the construction of Cantor's familiar "middle thirds" set. Define $K_0 = [0,1]$ and inductively define $K_n \subset K_{n-1}$ by removing an open interval of length $2(1-\varepsilon)2^{-2n}$. By the construction each K_n has 2^n connected components with length a_n which satisfy

$$\begin{cases} a_n = \frac{1}{2}(a_{n-1} - 2\varepsilon 2^{-2n}), \quad n = 1, 2, \ldots \\ a_0 = 1, \end{cases}$$

from which we get $a_n = (1-\varepsilon)2^{-n} + \varepsilon 2^{-2n}$. Thus

$$\mathcal{L}^1(K) = \lim_{n\to\infty} \mathcal{L}^1(K_n) = \lim_{n\to\infty} 2^n a_n = 1 - \varepsilon.$$

Take $G = [0,1] \setminus K$, then $\mathcal{L}^1(G) = \varepsilon$. On the other hand, G is dense in $[0,1]$ since the interior of K is empty.

(3) Here we construct a Cantor-like set, or a Cantor set with positive measure, with positive measure by modifying the construction of the Cantor set as follows. Let $\{a_k\}$ be a sequence of positive numbers satisfying

$$\gamma \equiv \sum_{k=1}^{\infty} 2^{k-1} a_k < 1.$$

Construct the set $\mathcal{S}$ so that at the kth stage of the construction one removes 2^{k-1} centrally situated open intervals each of length a_k. Establish the facts:

(a) $\mathcal{L}^1(\mathcal{S}) = 1 - \gamma$,

(b) $\mathcal{S}$ is perfect,

(c) $\mathcal{S}$ is uncountable.

Solution:

(a) As the intervals removed at the same stage or different stages are mutually disjoint, we have

$$\begin{aligned}\mathcal{L}^1(\mathcal{S}) &= 1-\sum_{k=1}^{\infty} 2^{k-1} \text{ length of interval removed in the k th stage} \\ &= 1-\sum_{k=1}^{\infty} 2^{k-1} a_k \\ &= 1-\gamma.\end{aligned}$$

Define $S_0 = [0,1]$ and inductively define $S_n \subset S_{n-1}$ by removing an open interval of length a_n. Obviously by the construction each S_n has 2^n connected components with length b_n which satisfy

$$\begin{cases} b_n = \frac{1}{2}(b_{n-1}-a_n), \quad n = 1, 2, \ldots \\ b_0 = 1, \end{cases}$$

By our construction of S_n, b_n is a non-negative decreasing sequence, so $\lim_{n\to\infty} b_n$ exist. $\gamma < \infty \Rightarrow a_n \to 0$. Therefore

$$\lim_{n\to\infty} b_n = 0.$$

(b) If $x \in S$, then x belongs some connected component of $S_n, \forall n \in \mathbb{N}$. Observe that the end points of the 2^n intervals of S_n are in S, so $\exists y_n$ end point of one of the interval s.t.

$$|y_n - x| \le b_n \to 0 \text{ as } n \to \infty.$$

We have S is a perfect set.

(c) We try to prove by contradiction. Suppose S is countable and let s_n be an enumeration of the set. s_1 belongs to exactly one of the two intervals in S_1, denote the interval which fails to contain s_1 by F_1. In S_2, 2 disjoint intervals $\subseteq F_1$ are obtained by removing an central interval of length a_2 from F_1, one of them say F_2 must fail to contain s_2. Repeating the process, we have decreasing sequence of closed interval F_n of length b_n

s.t s_n does not belongs to F_n and

$$\phi \neq \bigcap_{n=1}^{\infty} F_n \subseteq S.$$

Hence

$$\exists s \in S \text{ and } s \neq s_n, \forall n \in \mathbb{N}.$$

We have a contradiction.

(4) Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}^1(A)$.

Solution: Let $B = \{0,1,2,3,5,6,7,8,9\}$, the set $F_0 = \{x \in [0,1] : x = 0.4a_2a_3\cdots, a_j = 0,1,2,\cdots,9\} = [\frac{4}{10}, \frac{5}{10}]$ is of Lebesgue measure $\frac{1}{10}$. Fix $y_1 \in B$, $|B| = 9^1 = 9$, the set $F_{y_1} = \{x \in [0,1] : x = 0.y_1 4a_3\cdots, a_j = 0,1,2,\cdots,9\, \forall j \geq 3\} = [\frac{y_1}{10} + \frac{4}{100}, \frac{y_1}{10} + \frac{5}{100}]$ is of Lebesgues measure $\frac{1}{100}$. Fix $(y_1, y_2) \in B^2$, $|B^2| = 9^2 = 81$, the set $F_{(y_1,y_2)} = \{x \in [0,1] : x = 0.y_1y_2 4a_4\cdots, a_j = 0,1,2,\cdots,9\, \forall j \geq 4\}$ is of measure $\frac{1}{1000}$. Continuing the process, we have

$$A = [0,1] \setminus (\bigcup_{n=1}^{\infty} \bigcup_{(y_1,y_2,\cdots,y_n)\in B^n} F_{(y_1,y_2,\cdots,y_n)} \cup F_0)$$

and as all $F_{(y_1,y_2,\cdots,y_n)}, F_0$ are disjoint, we have

$$\begin{aligned} \mathcal{L}^1(A) &= 1 - \frac{1}{10} - \sum_{n=1}^{\infty} \sum_{(y_1,y_2,\cdots,y_n)\in B^n} \frac{1}{10^{n+1}} \\ &= 1 - \frac{1}{10} - \sum_{n=1}^{\infty} \frac{9^n}{10^{n+1}} \\ &= 0. \end{aligned}$$

(5) Let $\mathcal{N}$ be a Vitali set in $[0,1]$. Show that $\mathcal{M} = [0,1] \setminus \mathcal{N}$ has measure 1 and hence deduce that

$$\mathcal{L}^1(\mathcal{N}) + \mathcal{L}^1(\mathcal{M}) > \mathcal{L}^1(\mathcal{N} \cup \mathcal{M}).$$

Remark: I have no idea what $\mathcal{L}^1(\mathcal{N})$ is, except that it is positive.

Solution: We first prove that every Lebesgue measurable subset of $\mathcal{N}$ must be of measure zero. Let A be a Lebesgue measurable subset of $\mathcal{N}$, $\{A+q\}_{q\in\mathbb{Q}\cap[0,1)}$ is a sequence of disjoint measurable set contained inside $[-1,2]$. By translational invariance of Lebesgue measure,

$$\mathcal{L}^1(\bigcup_{q\in\mathbb{Q}\cap[0,1)} A+q) = \sum_{q\in\mathbb{Q}\cap[0,1)} \mathcal{L}^1(A+q) = \sum_{q\in\mathbb{Q}\cap[0,1)} \mathcal{L}^1(A) < \infty,$$

Therefore we must have

$$\mathcal{L}^1(A) = 0.$$

We try to prove by contradiction, suppose there is an open set G s.t. $\mathcal{L}^1(G) = 1-\varepsilon < 1$ and $G \supseteq \mathcal{N}^c$. Then $[0,1]\setminus G$ is a measurable subset of $\mathcal{N}$ satisfying

$$0 < \varepsilon = \mathcal{L}^1([0,1]) - \mathcal{L}^1(G) \le \mathcal{L}^1([0,1]\setminus G).$$

Contradicting to our previous result.

(6) Construct a Borel set $A \subset \mathbb{R}$ such that

$$0 < \mathcal{L}^1(A\cap I) < \mathcal{L}^1(I)$$

for every non-empty segment I. Is it possible to have $\mathcal{L}^1(A) < \infty$ for such a set?

Solution: Without loss of generality, we consider $n = 0$ and put $0 < c_0 = b < 1$. Choose a sequence $\varepsilon_m \searrow b$ with $b < \varepsilon_m < 1$. For $m = 1, 2, \ldots$, A_m is an open set constructed as follows:

(a) A_1 is the open dense set of $[0,1]$ of measure ε_1 as in (5).

(b) Each A_m is the union of countably many disjoint open intervals $I_{m,k}$ of length $\ell_{m,k}$, $k = 1, 2, \ldots$ with $\ell_{m,k} = 2(1-\varepsilon_m)2^{-2n}$ and $\sum_{k=1}^{\infty} \ell_{m,k} = \varepsilon_m$ as in (5).

(c) Having chosen A_m, $A_{m+1} \subset A_m$ is chosen such that $A_{m+1}\cap I_{m,k}$ is open, dense in $I_{m,k}$ and $\mathcal{L}^1(A_{m+1}\cap I_{m,k}) = \frac{\varepsilon_{m+1}}{\varepsilon_m}\ell_{m,k} < \ell_{m,k}$.

Note that $\mathcal{L}^1(A_{m+1}) = \frac{\varepsilon_{m+1}}{\varepsilon_m}\mathcal{L}^1(A_m) = \varepsilon_{m+1}$, showing that (b) is valid with m replaced by $m+1$.

Let $A = \bigcap_{m=1}^{\infty} A_m$ and $I = (a-\delta, a+\delta) \subset [0,1]$. To show that $0 < \mathcal{L}^1(A\cap I) < \mathcal{L}^1(I)$, we only need to find an open interval $I_{n_0,k_0} \subset I$ such that

$$\mathcal{L}^1(I_{n_0,k_0} \cap A) > 0 \quad .$$

Let n_0 satisfy $2^{-2n_0} < \frac{\delta}{4}$. Because A_{n_0} is dense in $[0,1]$, there exists a point $p \in A_{n_0}$ such that $|p-a| < \frac{\delta}{2}$. By (b), we can pick $k_0 \in \mathbb{N}$ such that $p \in I_{n_0,k_0}$. Since $\ell_{n_0,k_0} \leq 2^{-2n_0+1} < \frac{\delta}{2}$, we get $I_{n_0,k_0} \subset I$. Moreover, by (c), we obtain

$$\mathcal{L}^1(I_{n_0,k_0} \cap A_n) = \frac{\varepsilon_n}{\varepsilon_{n_0}} \ell_{n_0,k_0} \quad \text{and} \quad \mathcal{L}^1(I_{n_0,k_0} \setminus A_n) = \left(1 - \frac{\varepsilon_n}{\varepsilon_{n_0}}\right) \ell_{n_0,k_0}$$

for all $n \geq n_0$. Taking $n \to \infty$, we have

$$\mathcal{L}^1(I_{n_0,k_0} \cap A) = \frac{c_0}{\varepsilon_{n_0}} \ell_{n_0,k_0} > 0 \quad \text{and} \quad \mathcal{L}^1(I_{n_0,k_0} \setminus A) = \left(1 - \frac{c_0}{\varepsilon_{n_0}}\right) \ell_{n_0,k_0} > 0.$$

Finally, we can construct in each $[n, n+1]$, a Borel set A_n with $\mathcal{L}^1(A_n) = c_n$ with $0 < c_n < 1$ such that for every open interval $I \subset [n, n+1]$, $0 < \mathcal{L}^1(A \cap I) < \mathcal{L}^1(I)$ and

$$\sum_{n=-\infty}^{\infty} c_n = c < \infty.$$

If so, let $A = \bigcup_{n=-\infty}^{\infty} A_n$, then A satisfies the condition in the problem and $\mathcal{L}^1(A) = c < \infty$.

(7) Let E be a subset of $\mathbb{R}$ with positive Lebsegue measure. Prove that for each $\alpha \in (0,1)$, there exists an open interval I so that

$$\mathcal{L}^1(E \cap I) \geq \alpha \mathcal{L}^1(I).$$

It shows that E contains almost a whole interval. Hint: Choose an open G containing E such that $\mathcal{L}^1(E) \geq \alpha \mathcal{L}^1(G)$ and note that G can be decomposed into disjoint union of open intervals. One of these intervals satisfies our requirement.

Solution: As $\exists n \in \mathbb{N}$ s.t. $\mathcal{L}^1(E \cap (-n, n)) > 0$, WLOG we may assume that E has finite outer measure, then $\forall\alpha \in (0,1)$, $\exists$ open G s.t.$G \supseteq E$ and

$$\mathcal{L}^1(E) + \frac{(1-\alpha)}{\alpha}\mathcal{L}^1(E) \geq \mathcal{L}^1(G),$$

Hence

$$\mathcal{L}^1(E) \geq \alpha\mathcal{L}^1(G).$$

we can write $G = \bigcup_{i=1}^{\infty} I_i$ where I_i are disjoint open intervals. Then one of these I_i must satisfy the desired property, otherwise

$$\mathcal{L}^1(E) \leq \sum_{i=1}^{\infty} \mathcal{L}^1(E \cap I_i) < \alpha \sum_{i=1}^{\infty} \mathcal{L}^1(I_i) = \alpha\mathcal{L}^1(G) < \infty,$$

contradicting the above inequality.

(8) Let E be a measurable set in $\mathbb{R}$ with respect to $\mathcal{L}^1$ and $\mathcal{L}^1(E) > 0$. Show that $E - E$ contains an interval $(-a, a)$, $a > 0$. Hint:

(a) U, V open, with finite measure, $x \mapsto \mathcal{L}^1((x+U) \cap V)$ is continuous on $\mathbb{R}$.

(b) A, B measurable, $\mu(A), \mu(B) < \infty$, then $x \mapsto \mathcal{L}^1((x+A) \cap B)$ is continuous. For $A \subset U$, $B \subset V$, try

$$\mathcal{L}^1((x+U) \cap V) - \mathcal{L}^1((x+A) \cap B)| \leq \mathcal{L}^1(U \setminus A) + \mathcal{L}^1(V \subset B).$$

(c) Finally, $x \mapsto \mathcal{L}^1((x+E) \cap E)$ is positive at 0 and if $(x+E) \cap E \neq \emptyset$, then $x \in E \setminus E$.

Solution:

(a) We prove the case when U is an open interval I, note for all subset A, B of $\mathbb{R}$,

$$((x+A) \cap B) \setminus (((y+A) \cap B)) = (x+A) \setminus (y+A) \cap B.$$

Therefore

$$\left|\mathcal{L}^1((x+I)\cap V) - \mathcal{L}^1((y+I)\cap V)\right| \leq \mathcal{L}^1((x+I)\setminus(y+I)) + \mathcal{L}^1((y+I)\setminus(x+I)) \leq 4\left|x-y\right|.$$

the function is Lipschitz and continuous. In general U can be written as countable union of disjoint open intervals $\{I_i\}$, as $\sum_{i=1}^{\infty} \ell(I_i) < \infty, \exists N$ s.t. for all $k \geq N$,

$$\sum_{i=k}^{\infty} \ell(I_i) < \varepsilon.$$

We have

$$\sum_{i=1}^{\infty} \mathcal{L}^1((x+I_i)\cap V) - \mathcal{L}^1((y+I_i)\cap V) \leq \sum_{i=1}^{k} \mathcal{L}^1((x+I_i)\cap V) - \mathcal{L}^1((y+I_i)\cap V) + 2\varepsilon < 3\varepsilon$$

for x sufficiently close to y. Similarly

$$\sum_{i=1}^{\infty} \mathcal{L}^1((y+I_i)\cap V) - \mathcal{L}^1((x+I_i)\cap V) \leq 3\varepsilon.$$

We have the function $\mathcal{L}^1((x+U)\cap V)$ is continuous.

(b) Obviously , $((x+U)\cap V)\setminus((x+A)\cap B) \subseteq U\setminus A \cup V\setminus B$. Therefore, we have

$$0 \leq \mathcal{L}^1((x+U)\cap V) - \mathcal{L}^1((x+A)\cap B) \leq \mathcal{L}^1(U\setminus A) + \mathcal{L}^1(V\setminus B).$$

Note RHS is independent on x, y, so the result follow from outer regularity of Lebesgue measure.

(c) the function $\mathcal{L}^1((x+E)\cap E)$ is continuous and positive at 0, $\exists a > 0$ s.t the function remain positive on $(-a, a)$, i.e

$$(x+E)\cap E \neq \emptyset$$

and $\forall x \in (-a, a), \exists e_1 e_2 \in E$ s.t

$$x = e_1 - e_2 \in E - E.$$

Alternate proof. The following is a simple proof due to Karl Stromberg.

By the regularity of $\mathcal{L}^1$, for every $\varepsilon > 0$ there are a compact set $K \subset E$ and an open set $U \supset E$ such that

$$\mathcal{L}^1(K) + \varepsilon > \mathcal{L}^1(E) > \mathcal{L}^1(U) - \varepsilon.$$

For our purpose it is enough to choose K and U such that

$$2\mathcal{L}^1(K) > \mathcal{L}^1(U).$$

Since $K \subset U$, there is an open cover of K that is contained in U. Since K is compact, one can choose a small neighborhood V of 0 such that

$$K + V \subset U.$$

Let $v \in V$, and suppose

$$(K + v) \cap K = \emptyset.$$

Then,

$$2\mathcal{L}^1(K) = \mathcal{L}^1(K + v) + \mathcal{L}^1(K) < \mathcal{L}^1(U),$$

contradicting our choice of K and U. Hence for all $v \in V$ there exists $k_1, k_2 \in K \subset E$ such that

$$k_1 + v = k_2,$$

which means that $V \subset E - E$.

(9) Give an example of a continuous map ϕ and a measurable f such that $f \circ \phi$ is not measurable. Hint: May use the function $h = x + g(x)$ where g is the Cantor function as ϕ.

Solution: Let $h = x + g(x)$ where g is the Cantor function. Then $h : [0, 1] \to [0, 2]$ is a strictly monotonic and continuous map, so its inverse $\phi = h^{-1}$ is continuous too. Since g is constant on every interval in the complement of C, one has that h maps such an interval to an interval of the same length. Hence $\mu(h(C)) = 1$, where C is the cantor set. Then $h(C)$ contains a non-measurable set A due to Proposition 3.3. Let $B = \phi(A)$. Set $f = \chi_B$. Then $f \circ \phi$ is not measurable since its inverse image of 1 is A.