MATH4240 Tutorial 6 Notes

1 Stationary Distribution

For a Markov chain X, a probability distribution π on the state space S is a *stationary distribution* if it is invariant under the process: if X_0 has distribution π, then X_n also has distribution π for all n. If P is the transition function, this is $\pi(y) = (\pi P)(y) = \sum_x \pi(x)P(x,y)$.[1] If the state space is finite, π is the *left* eigenvector of P associated to the eigenvalue 1 (which is always an eigenvalue of P).

Recall the following theorem from lecture:

Theorem 1.1. *If P is the transition matrix of a finite Markov chain, and*

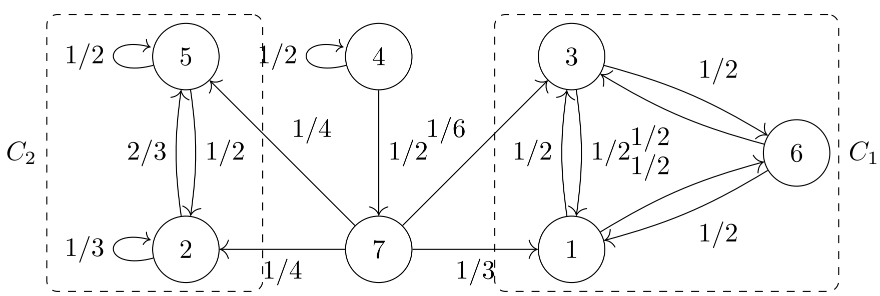
- 1 *is a simple eigenvalue of P*
- *no other (complex) eigenvalue λ has modulus $|\lambda| \geq 1$ (i.e., 1 is dominant eigenavalue)*
- *there exists a **left** eigenvector π associated to the eigenvalue 1 that has nonnegative entries (which we will also assume to be normalized such that $\sum \pi_i = 1$)*

Then $\lim_n P^n$ exists and is of the form $\lim_n P^n = \begin{pmatrix}\pi & \dots & \pi\end{pmatrix}^{\mathsf{T}}$ (that is, π stacked together vertically), and π is the unique stationary distribution.

As mentioned in the lecture, these conditions are satisfied if for some n every entry of P^n is positive ("*primitive*").[2]

2 Computational Example

Recall the Markov chain from tutorial 4:



which, as we have computed, can be condensed into an absorbing chain that has a limiting transition matrix

$$\tilde{P} = \begin{array}{c} \begin{matrix} C_1 & C_2 & 4 & 7 \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{array}\right] \end{array} = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}, \qquad \lim_n \tilde{P}^n = \begin{array}{c} \begin{matrix} C_1 & C_2 & 4 & 7 \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{array}\right] \end{array} = \begin{pmatrix} I & 0 \\ NR & 0 \end{pmatrix}$$

[1] Note that we are summing on the *first* variable.

[2] If the chain has N states, it suffices to take $n \geq (N-1)^2 + 1$, see Horn and Johnson. This means you can check primitivity by just squaring the transition matrix a few times and checking if there is a zero entry. Still, you would need to compute the left eigenvector.

with fundamental matrix $N = (I - Q)^{-1}$.

To find the limiting transition matrix of *the original chain*, it suffices to consider the chain on each of the irreducible closed sets. For convenience, let us first permute the transition matrix (of the original chain) into canonical form

$$P_{\text{can}} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{ccc|cc|cc} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/6 & 0 & 1/4 & 1/4 & 0 & 0 \end{array}\right] \end{array}$$

For $C_1 = \{1, 3, 6\}$, the transition matrix restricted on C_1 is

$$P_{C_1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

With direct computations, P_{C_1} has eigenvalues $1, -1/2, -1/2$, and the left eigenvector of P_{C_1} associated to eigenvalue 1 is of the form $\pi = c(1, 1, 1)$ for $c \in \mathbb{R}$. These imply Theorem 1.1 holds[3], and so with normalized left eigenvector $\pi_{C_1} = (1/3, 1/3, 1/3)$,

$$\lim_n P_{C_1}^n = \begin{pmatrix} \pi_{C_1} \\ \pi_{C_1} \\ \pi_{C_1} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \vec{1}\pi_{C_1}$$

Here, $\vec{1}$ is a column vector of appropriate size in which every entry is 1.

For $C_2 = \{2, 5\}$, the transition matrix restricted on C_2 is

$$P_{C_2} = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$$

With direct computations, P_{C_2} has eigenvalues $1, -1/6$, and the left eigenvector of P_{C_2} associated to eigenvalue 1 is of the form $\pi = c(3, 4)$ for $c \in \mathbb{R}$. These imply Theorem 1.1 holds, and so with normalized left eigenvector $\pi_{C_2} = (3/7, 4/7)$,

$$\lim_n P_{C_2}^n = \begin{pmatrix} \pi_{C_2} \\ \pi_{C_2} \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{pmatrix} = \vec{1}\pi_{C_2}$$

with $\vec{1}$ similarly defined.

These imply that the limiting transition matrix exists and takes the form noted in lecture.

We now need to find the part of the limit transition matrix from the transient states to the irreducible closed sets $\lim_n P^n(S_T, C_i)$. As covered in the lecture, this part is simply

$$\begin{pmatrix} \rho(4, C_1)\pi_{C_1} & \rho(4, C_2)\pi_{C_2} \\ \rho(7, C_1)\pi_{C_1} & \rho(7, C_2)\pi_{C_2} \end{pmatrix} = \begin{array}{c} \\ 4 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 6 & 2 & 5 \end{array} \\ \left[\begin{array}{ccc|cc} 1/6 & 1/6 & 1/6 & 3/14 & 2/7 \\ 1/6 & 1/6 & 1/6 & 3/14 & 2/7 \end{array}\right] \end{array}$$

Note that we can *abuse the notation* and write it in a more appealing form

$$\begin{pmatrix} \rho(4, C_1)\pi_{C_1} & \rho(4, C_2)\pi_{C_2} \\ \rho(7, C_1)\pi_{C_1} & \rho(7, C_2)\pi_{C_2} \end{pmatrix} = \begin{pmatrix} \rho(4, C_1) & \rho(4, C_2) \\ \rho(7, C_1) & \rho(7, C_2) \end{pmatrix} \begin{pmatrix} \pi_{C_1} & 0 \\ 0 & \pi_{C_2} \end{pmatrix} = NR \begin{pmatrix} \pi_{C_1} & 0 \\ 0 & \pi_{C_2} \end{pmatrix}$$

[3] You can also note that P_{C_1} is primitive: $P_{C_1}^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$.

So, collecting the results, the limiting transition matrix for the original chain, *in the canonical form*, is

$$\lim_n P^n_{\text{can}} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{c|c|c} \lim_n P^n_{C_1} & 0 & 0 \\ \hline 0 & \lim_n P^n_{C_2} & 0 \\ \hline \begin{matrix}\rho(4,C_1)\pi_{C_1} \\ \rho(7,C_1)\pi_{C_1}\end{matrix} & \begin{matrix}\rho(4,C_2)\pi_{C_2} \\ \rho(7,C_2)\pi_{C_2}\end{matrix} & 0 \end{array}\right] \end{array} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{ccc|cc|cc} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ \hline 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \end{array}\right] \end{array}$$

We can *abuse the notation* and write it in a form that is a bit more intuitive:

$$\lim_n P^n_{\text{can}} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \left[\begin{array}{c|c|c} \begin{matrix}\pi_{C_1} \\ \pi_{C_1} \\ \pi_{C_1}\end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix}\pi_{C_2} \\ \pi_{C_2}\end{matrix} & 0 \\ \hline \multicolumn{2}{c|}{NR\begin{pmatrix}\pi_{C_1} & 0 \\ 0 & \pi_{C_2}\end{pmatrix}} & 0 \end{array}\right] \end{array} = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & S_T \end{array} \\ \left[\begin{array}{cc|c} \vec{1} & 0 & 0 \\ 0 & \vec{1} & \\ \hline \multicolumn{2}{c|}{NR} & 0 \end{array}\right] \end{array} \begin{array}{c} C_1 \\ C_2 \\ S_T \end{array} \left[\begin{array}{cc|c} \pi_{C_1} & 0 & 0 \\ 0 & \pi_{C_2} & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right]$$

which you can interpret as total probability on the stationary distributions, conditioning on which irreducible closed set the chain ends up in. Note that the $\vec{1}$ vectors should have appropriate sizes that match the corresponding irreducible closed sets.

Now we have the limiting transition matrix *in canonical form*. Typically you are given the transition matrix with the states in a certain order, so you should permute the states back to the original order.

$$\lim_n P^n = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \end{bmatrix} \end{array}$$

2.1 Example from Lecture

I believe the following transition matrix is mentioned during the lecture (for convenience, we also label the states)

$$P = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \left[\begin{array}{cc|c|ccc} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \end{array}$$

Let us compute the limiting transition matrix. Note that the matrix is already in canonical form.

The condensed chain, with $C_1 = \{1, 2\}, C_3 = \{3\}$ has transition matrix

$$\tilde{P} = \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \end{array} = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

For the (restricted) transition matrix $P_1 = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$ on C_1, which is the same one as in the previous example, the eigenvalues are $1, -1/6$, with the eigenvectors associated to the eigenvalue 1 being in the form $\pi = c(3,4)$ with $c \in \mathbb{R}$. By Theorem 1.1, on (normalized) stationary distribution $\pi_1 = (3/7, 4/7)$ on C_1, $\lim_n P_1^n = \begin{pmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{pmatrix}$.

On $C_2 = \{3\}$, it is easy to see that $\pi_2 = (1)$ is the stationary distribution on C_2, and $\lim_n P_2^n = \begin{pmatrix} 1 \end{pmatrix}$.

To compute the transient part, we can compute[4]

$$N = (I-Q)^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}, \quad NR = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}, \quad NR \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} = \begin{pmatrix} 9/28 & 3/7 & 1/4 \\ 3/14 & 2/7 & 1/2 \\ 3/28 & 1/7 & 3/4 \end{pmatrix}$$

Combined,

$$\lim_n P^n = \left[\begin{array}{c|c|ccc} \pi_1 & 0 & 0 & 0 & 0 \\ \pi_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \pi_2 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{NR\begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}} & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 9/28 & 3/7 & 1/4 & 0 & 0 & 0 \\ 3/14 & 2/7 & 1/2 & 0 & 0 & 0 \\ 3/28 & 1/7 & 3/4 & 0 & 0 & 0 \end{bmatrix}$$

[4] If we denote $T = NR\Pi$ with $\Pi = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}$, then T can be solved from the equation $(I-Q)T = R\Pi$, which may be a bit easier to handle (as you do not need to compute the inverse N explicitly).