MATH4240 Tutorial 3 Notes

1 One-step Argument

Given a Markov chain X_n with (finite) state space S, there are a few things we may want to know on given states $x, y \in S$:

- the distribution of T_y on initial state x: $P_x(T_y = n)$
- probability of visit: $\rho_{xy} = P_x(T_y < \infty) = \sum_n P_x(T_y = n)$, which also gives expected number of visit $E_x(N(y)) = \frac{\rho_{xy}}{1-\rho_{yy}}$, with $N(y) = \sum_{n\geq 1} \chi_y(X_n)$
- expected hitting time: $E_x(T_y) = \sum_n n P_x(T_y = n)$

(Recall that $T_y = \inf\{\, n \geq 1 \mid X_n = y \,\}$ is the hitting time of state y *after* the initial state.)

One-step argument yields a linear system on these quantities by considering their behavior after one step[1] in the chain: (for simplicity, let us denote for the moment $t_{xy} = E_x(T_y \,|\, T_y < \infty)$)

$$\begin{aligned}
P_x(T_y = n+1) &= 0 && + \sum_{z\neq y} P(x,z) P_z(T_y = n) \quad \text{for } n \geq 1 \\
P_x(T_y = 1) &= P(x,y) && \\
\rho_{xy} &= P(x,y) && + \sum_{z\neq y} P(x,z)\rho_{zy} \\
t_{xy} &= P(x,y)\cdot 1 && + \sum_{z\neq y} P(x,z)(1 + t_{zy}) \\
&= 1 && + \sum_{z\neq y} P(x,z) t_{zy}
\end{aligned}$$

If $T_y = \infty$ happens with probability $1 - \rho_{xy} > 0$ (e.g. not *irreducible*), then trivially $E_x(T_y) = \infty$. Note that the equations still hold if we consider the hitting time of a set T_A instead of a single state T_y (with obvious modification).

We can collect the equations on ρ_{xy} and t_{xy} as matrices:

$$\begin{pmatrix} \rho_{1y} \\ \vdots \\ \rho_{Ny} \end{pmatrix} = \begin{pmatrix} P(1,y) \\ \vdots \\ P(N,y) \end{pmatrix} + P_{-y} \begin{pmatrix} \rho_{1y} \\ \vdots \\ \rho_{Ny} \end{pmatrix}$$

$$\begin{pmatrix} t_{1y} \\ \vdots \\ t_{Ny} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + P_{-y} \begin{pmatrix} t_{1y} \\ \vdots \\ t_{Ny} \end{pmatrix}$$

if the states are denoted as $1, \ldots, N$, and $(P_{-y})_{ij} = \begin{cases} P_{ij} & \text{if } j \neq y \\ 0 & \text{if } j = y \end{cases}$ is the the transition matrix with the column corresponding to state y replaced with 0.

While you *can* solve these systems with straightforward approaches from e.g. MATH1030, sometimes the transition matrix is dense enough that it *may* be worth while to find other approaches, especially if you just want to find a single specific ρ_{xy} (or t_{xy}).

[1] Instead of first-step analysis, you can also do last-step analysis.

2 Examples

Example 1. Consider Ehrenfest chain on population $d = 3$: on state space $S = \{ 0, \dots, d \}$,

$$P(x,y) = \begin{cases} x/d & \text{if } y = x-1 \\ 1 - x/d & \text{if } y = x+1 \\ 0 & \text{otherwise} \end{cases}, \quad \text{or} \quad P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

For each $x \in S$, what is ρ_{x0} and $E_x(T_0)$?

With one-step argument, we have the linear system

$$\begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{20} \\ \rho_{30} \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{20} \\ \rho_{30} \end{pmatrix}$$

$$\begin{pmatrix} E_0(T_0) \\ E_1(T_0) \\ E_2(T_0) \\ E_3(T_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_0(T_0) \\ E_1(T_0) \\ E_2(T_0) \\ E_3(T_0) \end{pmatrix}$$

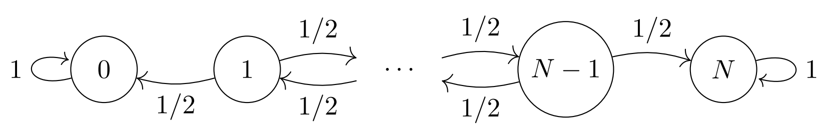
Solving the system (e.g. with Gauss elimination, reducing variables) yields

$$\begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{20} \\ \rho_{30} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} E_0(T_0) \\ E_1(T_0) \\ E_2(T_0) \\ E_3(T_0) \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 9 \\ 10 \end{pmatrix}$$

Example 2. Let us consider the (fair) gambler's ruin chain on state space $S = \{ 0, \dots, N \}$:

$$P(i, i-1) = P(i, i+1) = \frac{1}{2} \quad \text{if } i \neq 0, N$$
$$P(0,0) = P(N,N) = 1$$

or as transition diagram



Let us compute the ruin probability ρ_{x0} when started at state $x \neq 0, N$.

By one-step argument and noting that $\rho_{00} = 1$, $\rho_{N0} = 0$,

$$\rho_{x0} = P(x,0) + \sum_{z \neq 0} P(x,z)\rho_{z0}$$

$$\text{so} \quad \rho_{10} = \frac{1}{2} + \frac{1}{2}\rho_{20}$$
$$\rho_{i0} = \frac{1}{2}\rho_{i-1,0} + \frac{1}{2}\rho_{i+1,0} \quad \text{if } i \in \{ 2, \dots, N-1 \}$$

Moving the terms around, we have

$$\rho_{20} - \rho_{10} = \rho_{10} - 1$$
$$\rho_{i+1,0} - \rho_{i0} = \rho_{i0} - \rho_{i-1,0} \quad \text{if } i \in \{ 2, \dots, N-1 \}$$

which implies for $i \in \{ 1, \dots, N \}$,

$$\begin{aligned} \rho_{i0} - \rho_{i-1,0} &= \rho_{10} - 1 \\ \rho_{i0} - 1 &= \sum_{j=1}^{i} (\rho_{j0} - \rho_{j-1,0}) = i(\rho_{10} - 1) \end{aligned}$$

In particular, $-1 = \rho_{N0} - 1 = N(\rho_{10} - 1)$. So $\rho_{i0} = 1 - i/N$. Same approach also implies $\rho_{iN} = i/N$.

We can also compute the expected duration $E_x(T)$ of the chain, where $T = T_{\{0,N\}} = \min(T_0, T_n)$ is the (random) time of absorption, again when started at $x \neq 0, N$.

By one-step argument,

$$\begin{aligned} E_x(T) &= 1 + \sum_{z \notin \{0,N\}} P(x,z) E_z(T) \\ \text{so} \quad E_1(T) &= 1 + \frac{1}{2} E_2(T) \\ E_{N-1}(T) &= 1 + \frac{1}{2} E_{N-2}(T) \\ E_i(T) &= 1 + \frac{1}{2} E_{i-1}(T) + \frac{1}{2} E_{i+1}(T) \quad \text{if } i \in \{ 2, \dots, N-2 \} \end{aligned}$$

Moving the terms around and denoting for the moment $E_0(T) = E_N(T) = 0$, we have for each $i \in \{ 0, \dots, N-1 \}$,

$$E_{i+1}(T) - E_i(T) = E_i(T) - E_{i-1}(T) - 2$$

which implies for $i \in \{ 1, \dots, N \}$,

$$\begin{aligned} E_i(T) - E_{i-1}(T) &= E_1(T) - 2(i-1) \\ E_i(T) &= \sum_{j=1}^{i} E_j(T) - E_{j-1}(T) = iE_1(T) - i(i-1) \end{aligned}$$

In particular, $0 = E_N(T) = NE_i(T) - N(N-1)$. So, $E_1(T) = N - 1$, and hence

$$E_x(T) = x(N-x)$$

Example 3. Let us consider symmetric random walk on $S = \mathbb{Z}$:

$$P(x, x+1) = P(x, x-1) = \frac{1}{2} \quad \text{for } x \in \mathbb{Z}$$

What is the probability ρ_{x0} of visiting state 0 when started at state $x \in \mathbb{Z}$?

Like gambler's ruin chain, we have a similar list of equations

$$\begin{aligned} \rho_{00} &= \frac{1}{2}\rho_{10} + \frac{1}{2}\rho_{-1,0} \\ \rho_{10} &= \frac{1}{2} + \frac{1}{2}\rho_{20} \\ \rho_{-1,0} &= \frac{1}{2} + \frac{1}{2}\rho_{-2,0} \\ \rho_{x0} &= \frac{1}{2}\rho_{x+1,0} + \frac{1}{2}\rho_{x-1,0} \quad \text{for } x \neq -1, 0, 1 \end{aligned}$$

so the same approach yields

$$\begin{aligned} \rho_{x0} &= \rho_{10} + (x-1)(\rho_{20} - \rho_{10}) \quad \text{for } x \geq 2 \\ \rho_{x0} &= \rho_{-1,0} + (x-1)(\rho_{-2,0} - \rho_{-1,0}) \quad \text{for } x \leq -2 \end{aligned}$$

Since $0 \leq \rho_{x0} \leq 1$ for all x, we must have $\rho_{x0} = \rho_{10}$ for $x \geq 1$ and $\rho_{x0} = \rho_{-1,0}$ for $x \leq -1$. Solving the remaining equations gives $\rho_{10} = \rho_{-1,0} = 1$, and thus $\rho_{00} = 1$.

This implies that 0 is a recurrent state, and (by symmetry) the symmetric random walk on $\mathbb{Z}$ recurrent.