MATH4240 Tutorial 10 Notes

1 Basics

- A random variable X (with value in $[0,\infty)$) is *memoryless* if $P(X > t+s \mid X > t) = P(X > s)$.

  If X is continuous, it must be exponentially distributed.

  If X is discrete, it must be geometrically distributed.

- If $\xi_i \sim \text{Exp}(\lambda_i)$ are independent, then $\min(\xi_1, \dots, \xi_n) \sim \text{Exp}(\sum \lambda_j)$ and $P(\min(\xi_1, \dots, \xi_n) = \xi_i) = \frac{\lambda_i}{\sum \lambda_j}$ (see Tutorial 1)

- For a Markov jump process, the transition function $P_{xy}(t)$ satisfies *Chapman–Kolmogorov equation*

$$
\begin{aligned}
P(t+s) &= P(t)P(s) \\
\text{that is,} \quad P_{t+s}(x,y) &= \sum_z P_t(x,z)P_s(z,y)
\end{aligned}
$$

- The *rate matrix / generator* $D = [q_{xy}] = P'(0) = \lim_{t\to 0^+} \frac{P(t)-I}{t}$ of a Markov jump process satisfies

  - $q_{xy} \geq 0$ for $x \neq y$ and $-q_x = q_{xx} \leq 0$
  - $\sum_y q_{xy} = 0$, or $q_x = \sum_{y\neq x} q_{xy}$
  - (forward equation) $P' = PD$, that is, $P'_{xy}(t) = \sum_z P_{xz}(t)D_{zy}$. In particular, $\pi' = \pi D$
  - (backward equation) $P' = DP$, that is, $P'_{xy}(t) = \sum_z D_{xz}P_{zy}(t)$

  In particular, $P(t) = \exp(tD)$.

  (Compare with discrete-time case where $D = P - I$)

- If x is not an absorbing state, then the transition probability Q_{xy} of the *embedded Markov chain / jump chain* for $x \neq y$ is

$$
Q_{xy} = q_{xy}/q_x = \frac{q_{xy}}{\sum_{z\neq x} q_{xz}}
$$

  And if x is absorbing, $Q_{xy} = 0$ for $x \neq y$ and $Q_{xx} = 1$.

- Recall that in last session we talked about an alternative definition of Poisson process:

  Definition 1. A monotone increasing random process X_t with natural number value is a Poisson process with rate λ if it has independent increment, $X_0 = 0$, and

$$
\begin{aligned}
P(X_{t+h} - X_t = 0) &= 1 - \lambda h + o(\ h\ ) \\
P(X_{t+h} - X_t = 1) &= \lambda h + o(\ h\ )
\end{aligned}
$$

  Taking $t = 0$ we can see that

$$
\begin{aligned}
\frac{1}{h}(P(X_{t+h} = X_t) - 1) &= -\lambda + o(\ 1\ ) \\
\frac{1}{h}(P(X_{t+h} = X_t + 1) - 0) &= \lambda + o(\ 1\ )
\end{aligned}
$$

which means that the rate matrix and the jump matrix are of the form

$$D = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

So to specify a Markov jump process, it suffices to specify the rate matrix.

Example 1. Suppose at time $t > 0$ a Poisson process X_t with rate λ has exactly n arrivals $X_t = n$. Let us compute the distribution of the arrival times $T_1, \dots, T_n$, that is the pdf $f(t_1, \dots, t_n \mid X_t = n)$. Easy to see that

$$\{ \, T_1 = t_1, \, \dots, T_n = t_n, \, X_t = n \, \} = \{ \, \xi_1 = t_1, \, \xi_n = t_n - t_{n-1}, \, \xi_{n+1} > t - t_n \, \}$$

and by assumption, the waiting times are independent. These imply

$$\begin{aligned} f_{T_1,\dots,T_n}(t_1, \dots, t_n \mid X_t = n) &= f_{\xi_1}(t_1) \dots f_{\xi_n}(t_n - t_{n-1}) P\left(X_{n+1} > t - t_n\right) / P\left(X_t = n\right) \\ &= \frac{\lambda e^{-\lambda t_1} \dots \lambda e^{-\lambda(t_n - t_{n-1})} e^{-\lambda(t - \lambda_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\ &= n!/t^n \end{aligned}$$

for $0 \leq t_1 < \dots < t_n \leq t$. In particular, $(T_1, \dots, T_n \mid X_t = n)$ is uniformly distributed on $\{ \, 0 \leq t_1 < \dots < t_n \leq t \, \}$.

In the specific case $n = 1$, the arrival is uniformly distributed: $(T_1 \mid X_t = 1) \sim \text{Unif}(0, t)$.

2 Thinning

Recall that Poisson process satisfies the following two properties:

- (Superposition) The sum $\sum_i X_i(t)$ of independent Poisson process $X_i(t)\text{Poi}(\lambda_i t)$ is a Poisson process with rate being the sum of all individual rates:

$$\sum X_i(t) \sim \text{Poi}\left((\sum \lambda_i) t \right)$$

- (Thinning) If each arrival of a Poisson process $X(t) \sim \text{Poi}(\lambda t)$ independently has probability p_i of being type i, the arrival processes $Y_1(t), \dots, Y_n(t)$ of individual types are independent Poisson processes with weighted rate:

$$Y_i(t) \sim \text{Poi}(p_i \lambda t)$$

Example 2. Suppose we have two independent Poisson processes X_t, Y_t with rate λ, μ respectively. What is the probability that we observe k arrivals of X_t at the nth arrival of Y_t?

Let us consider the total process $Z_t = X_t + Y_t$, which is also a Poisson process with rate $\lambda + \mu$. Then X_t, Y_t can be considered as the thinning of Z_t, that is, each arrival of Z_t has probability $p = \frac{\lambda}{\lambda+\mu}$ of being an arrival for X_t (and $1 - p = \frac{\mu}{\lambda+\mu}$ for Y_t).

The event that we see k arrivals of X_t at the nth arrival of Y_t is the same as having $k - 1$ successes (for being arrival of X_t) in the first $k + n - 1$ arrivals, and the $(k + n)$th trial is a success (to be of X_t). So the probability is

$$\binom{k+n-1}{k-1} p^{k-1} (1-p)^n \cdot p = \binom{k+n-1}{n} (\frac{\mu}{\lambda+\mu})^n (\frac{\lambda}{\lambda+\mu})^k$$

In another word, the number of arrivals from Y given that we see k arrivals from X obeys negative binomial distribution $N \sim \text{NegBin}(k, \frac{\lambda}{\lambda+\mu})$.

Example 3. In a bank there are two tellers, Teller 1 and Teller 2, with exponentially distributed serving time with mean 3 and 6 minutes respectively. Suppose three people, A, B and C, enter the bank, and A, B are served by the tellers first while C waits for the first teller available.

1. What is the expected total amount of time for C to complete the service?
2. What is the expected time until the last one leave?

3. What is the probability that C is the last one to leave?

For simplicity, let us assume A is served by Teller 1, and B is served by Teller 2.
It is possible to consider the service time T_A, T_B, T_C of each person individually and compute respectively

1. $E\left(\min(T_A, T_B) + T_C\right)$
2. $E\left(\max(T_A, T_B, \min(T_A, T_B) + T_C)\right)$
3. $P\left(\max(T_A, T_B, \min(T_A, T_B) + T_C) = \min(T_A, T_B) + T_C\right)$

which you can handle by using the min property of exponential distribution, the identity $\max(x, y) = x + y - \min(x, y)$, and some algebra. (You should try to solve it this way.)

Here, we will use another approach. Instead of 3 people waiting, let us imagine there are infinitely many people waiting to be served, and A, B, C are the first three in the queue.

Let X_t, Y_t be the number of customers departed from the tellers (after being served) respectively. Then they must be Poisson processes $X_t \sim \text{Poi}(\frac{1}{3}t)$ and $Y_t \sim \text{Poi}(\frac{1}{6}t)$, so the total departure $Z_t = X_t + Y_t \sim \text{Poi}((\frac{1}{3} + \frac{1}{6})t) = \text{Poi}(\frac{1}{2}t)$ is also Poisson. This implies

1. On T_C being the service time of C, and τ_1 being the first arrival time of Z, we have

$$\begin{aligned}
&E\left(\text{time C complete service}\right) \\
&= E\left(\tau_1 + T_C\right) \\
&= E\left(\tau_1\right) + E\left(T_C \,|\, \text{first departure is from teller 1}\right) P\left(\text{first departure is from teller 1}\right) \\
&\qquad + E\left(T_C \,|\, \text{first departure is from teller 2}\right) P\left(\text{first departure is from teller 2}\right) \\
&= 2 + 3 \cdot \frac{1/3}{1/2} + 6 \cdot \frac{1/6}{1/2} = 6
\end{aligned}$$

2. Let τ_2 be the time of the second departure (from Z), and T be the time between the second departure and the last departure of A, B, C. Then $E\left(\text{time last one leaves}\right) = E\left(\tau_2 + T\right)$.

   By the memoryless property, we may assume that the last one to leave starts being served *exactly when the second departure happens.* Easy to see that the last one must be served by a different teller from the second one, so

$$\begin{aligned}
&E\left(\text{time last one leaves}\right) \\
&= E\left(\tau_2 + T\right) \\
&= 2 \cdot 2 + E\left(\text{teller 2 service time}\right) P\left(\text{second departure is from teller 1}\right) \\
&\qquad + E\left(\text{teller 1 service time}\right) P\left(\text{second departure is from teller 2}\right) \\
&= 2 \cdot 2 + 6 \cdot \frac{1/3}{1/2} + 3 \cdot \frac{1/6}{1/2} = 9
\end{aligned}$$

3. C is the last one to leave iff the first two departures are from different tellers. So

$$\begin{aligned}
&P\left(\text{C is last to leave}\right) \\
&= P\left(\text{first is from teller 1}\right) P\left(\text{second is from teller 2}\right) \\
&\qquad + P\left(\text{first is from teller 2}\right) P\left(\text{second is from teller 1}\right) \\
&= 2 \cdot \frac{1/3}{1/2} \frac{1/6}{1/2} = 4/9
\end{aligned}$$