THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2024/25 Term 2

Chapter II Stationary Distributions

1 SD and its computations

• Recall that for a MC $\{X_n\}_{n=0}^{\infty}$,

$$\vec{\pi}_{n+1} = \vec{\pi}_n P, \quad \vec{\pi}_n = \vec{\pi}_0 P^n, \quad n = 0, 1, 2, \cdots \tag{1}$$

where $\vec{\pi}_n$, $n \geq 0$, denote the p.d.f. of X_n.

• Consider a MC with P and S (for instance, $S = \{0, 1, 2, \cdots, N\}$ with N finite or infinite). $\vec{\pi} := [\pi(0), \pi(1), \cdots, \pi(N)]$, or denoted by $\pi(x)$, $x \in S$, is called a **stationary distribution** for P if

(i) $\vec{\pi}$ is a distribution, i.e., $\pi(x) \geq 0$, $\forall\, x \in S$, and $\sum_{x \in S} \pi(x) = 1$.

(ii) $\vec{\pi}$ is stationary: $\vec{\pi} P = \vec{\pi}$, i.e.,

$$\sum_{x \in S} \pi(x) P(x, y) = \pi(y), \quad \forall\, y \in S. \tag{2}$$

Here, (ii) means that if the chain starts from the distribution $\vec{\pi}$, then all X_n, $n \geq 1$ have the same distributions as $\vec{\pi}$.

• We have to notice:

(a) Given an initial distribution $\vec{\pi}_0$, if the limit distribution exists, i.e., $\lim_{n\to\infty} \vec{\pi}_0 P^n$ exists, denoted by $\vec{\pi}$, then $\vec{\pi}$ satisfies

$$\vec{\pi} = \left(\lim_{n \to \infty} \vec{\pi}_0 P^{n-1} \right) \cdot P = \vec{\pi} P, \tag{3}$$

i.e., the limit distribution $\vec{\pi}$ is stationary and hence $\vec{\pi}$ is a SD. Moreover, if $\vec{\pi} = \vec{\pi} P$ has a unique distribution solution then the limit distribution is independent of the initial distribution.

(b) If

$$\lim_{n \to \infty} P^n = \begin{pmatrix} \vec{\pi} \\ \vec{\pi} \\ \vdots \\ \vec{\pi} \end{pmatrix} \tag{4}$$

for some distribution $\vec{\pi}$, then the limit distribution exists and is independent of the initial distribution. We will discuss the long-term behavior of P^n in the last subsection.

• In case S is finite, we have some general conditions to assure the existence and uniqueness. In fact, let P be a Markov matrix with **finite** state space S. Assume

(i) the left 1-eigenvector (which must exist; *why?*) can be chosen to have all nonnegative entries.

(ii) 1 is a simple eigenvalue.

(iii) all other eigenvalues: $|\lambda_i| < 1$.

Then P has a unique SD $\vec{\pi}$, and (4) holds true. In particular, if for some n, P^n has all entries strictly positive, then three conditions above can be satisfied and the conclusion is true for the chain.

In the future lecture, we will show that *an irreducible MC with finite state space must have a unique SD* (but (4) may NOT hold true!).

• In the general situation that S is finite or infinite, we will discuss the existence and uniqueness of SD later on.

• Computation issues on SD, as well as the limit of P^n **if it exists**:

- In case S is finite and P is irreducible, apply Row Operators to $P^T - I$ to get the upper diagonal form.
- In case S is finite and P is reducible, apply the State Decomposition, for instance, $S = C_1 \cup C_2 \cup S_T$, re-write P as the canonical form, and then try to find the limit of P^n as $n \to \infty$, **if it exists**. See the tutorial and exercises for examples.
- In case S is infinite, see the lectures for two additional examples:
  (a) Find SD of an irreducible birth and death chain.
  (b) Find SD of a telephone exchange model with new calls satisfying the Poisson distribution (or a general queuing chain model with the service given by the rule that each person at the beginning of a unit time has the probability q to be served and leave the waiting line by the end of the unit time).

2 Average number of visits

• Given a MC with S and P, let $N_n(y)$ be the NO of visits to y in n-steps (i.e., during times $m = 1, 2, \cdots, n$). We are interested in determining

$$\frac{N_n(y)}{n}, \quad \frac{E_x(N_n(y))}{n}, \quad \text{as } n \to \infty. \tag{5}$$

Note:

(i) $\frac{N_n(y)}{n}$ is a r.v., denoting the **proportion of the first n units of time that the chain visits y**, and the limit of $\frac{N_n(y)}{n}$ as $n \to \infty$ (if exists) means the **average NO of visits to y (per unit time)** or the frequency that the chain visits y. We can compute $N_n(y)$ as

$$N_n(y) = \sum_{m=1}^{n} 1_y(X_m). \tag{6}$$

(ii) $\frac{E_x(N_n(y))}{n}$ is the **expected value of $\frac{N_n(y)}{n}$ for a chain starting from x**, and hence its limit value (if exists) means the **expected average NO of visits to y (per unit time)** or the **expected frequency** that the chain visits y. We can compute $E_x(N_n(y))$ as

$$E_x(N_n(y)) = \sum_{m=1}^{n} P^m(x, y). \tag{7}$$

Thus, to determine $\lim_{n\to\infty} \frac{E_x(N_n(y))}{n}$ is equivalent to determine

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} P^m(x, y)}{n}. \tag{8}$$

Note that it could occur that the above limit exists but $\lim_{n\to\infty} P^n(x, y)$ may not exist!

• In case y is transient, it is direct to see

$$\lim_{n\to\infty} N_n(y) = N(y) < \infty \text{ with prob } 1, \quad \lim_{n\to\infty} E_x(N_n(y)) = \frac{\rho_{xy}}{1 - \rho_{yy}} < \infty, \tag{9}$$

and hence

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = 0 \text{ with prob } 1, \quad \lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = 0. \tag{10}$$

This means that in the long run, the average NO of visits to a transient state is zero, and its expected value is also zero.

• In case y is recurrent, we can show the following result. For simplicity we consider an irreducible recurrent MC only. Then, for any $y \in S$,

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{m_y} \text{ with prob } 1, \quad \lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = \frac{1}{m_y}, \quad x \in S, \tag{11}$$

where $m_y := E_y(T_y)$ denotes the **mean return time to y for a chain starting from y**. m_y can be understood to be the **mean waiting time**. Thus, two limits mean that *the visit frequency and the waiting time are reciprocal to each other!!!* It is heuristically obvious; see the lectures for the rigorous proof.

3 Waiting time and existence of stationary distribution

• $0 < m_x := E_x(T_x) \leqq \infty$ for a recurrent state x. Note: If x is recurrent, then $P_x(T_x = \infty) = 0$ and $P_x(T_x < \infty) = 1$, so there is $k_0 \geq 1$ such that $P_x(T_x = k_0) > 0$, hence $m_x = E_x(T_x) = \sum_{k=1}^{\infty} kP_x(T_x = k) \geq k_0 P_x(T_x = k_0) > 0$.

• A recurrent state x is called **positive recurrent** if $(0 <)m_x < \infty$, and **null recurrent** if $m_x = \infty$. Thus, a positive recurrent state comes back *in finite waiting time*, and a null recurrent state comes back *very rarely*.

• We can also discuss communications between positive recurrent states. In fact, one can prove that *if a positive recurrent state x leads to y then y is also a positive recurrent state.*

Recall that an irreducible MC with finite state space is recurrent. One can further show that *an irreducible MC with finite state space does not admit any null recurrent state*, and hence it is positive recurrent.

Recall that given S and P, we have the state decomposition

$$S = S_R \cup S_T = (\cup_{i=1}^{k} C_i) \cup S_T, \tag{12}$$

where k can be finite or infinite. Then, for each i, C_i is either positive recurrent or null recurrent. Moreover, if C_i is finite, then C_i must be positive recurrent.

• The waiting time m_x of a recurrent state x, or the frequency $1/m_x$ of the chain visiting x, would be connected with the stationary solution of the chain. In fact, one can show that *an irreducible positive recurrent MC has a unique stationary distribution $\vec{\pi}$, given by*

$$\pi(x) = \frac{1}{m_x} \in (0, 1), \quad x \in S. \tag{13}$$

Notice that the theorem gives us a way to find the value of waiting time m_x of any state x. Here are a few immediate consequences:

(a) An irreducible MC with finite state space has a unique SD $\vec{\pi}$ with $\pi(x) = 1/m_x$, $x \in S$.

(b) We may further show that *if a general MC has no positive recurrent state (i.e., any state is either null recurrent or transient), then the chain has no SD.* Therefore, *for an irreducible MC, it has a SD if and only if it is positive recurrent.* Exercise: Apply it to determine if a general irreducible birth and death chain is either positive recurrent, or null recurrent, or transient; see the criterion discussed in the lecture.

(c) Let C be an irreducible closed set of positive recurrent states. Then, the chain has a unique SD $\vec{\pi}$ **concentrated on** C:

$$\pi(x) = \begin{cases} \dfrac{1}{m_x} & \text{if } x \in C, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Moreover, if $\{C_i\}_{i=1}^N$ is a collection of irreducible closed set consisting of positive recurrent states, then $\pi := \sum_{i=1}^N \lambda_i \pi_i$ is a SD of the chain for any $0 \le \lambda_i \le 1$ with $\sum_{i=1}^N \lambda_i = 1$, where each π_i is the unique SD concentrated on C_i.

4 Periodicity

• Recall that it could occur that the chain admits a SD but $\lim P^n$ does not exist (hence the long-term behavior of the chain seems unclear!). For instance,

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

The SD exists, given by $\vec{\pi} = [1/2, 1/2]$. For such P you can compute

$$P^{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{2m+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

Thus, $\lim P^n$ does not exist, but you can still determine the long-term behavior of the chain in the following way

$$\lim_{m\to\infty} P^{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lim_{m\to\infty} P^{2m+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

We can discuss such property by using the periodicity of the chain.

• The **period** d_x of a state $x \in S$ is defined by

$$d_x = g.c.d.\,\{n \ge 1 : P^n(x,x) > 0\}. \tag{18}$$

Note that d_x is a positive integer with $1 \le d_x \le \min\{n \ge 1 : P^n(x,x) > 0\}$. If $P(x,x) > 0$ then $d_x = 1$.

For the chain with P given by (15),

$$\{n \ge 1 : P^n(0,0) > 0\} = \{2, 4, 6, \cdots\} = \{n \ge 1 : P^n(1,1) > 0\}. \tag{19}$$

Thus,

$$d_0 = d_1 = 2. \tag{20}$$

• For an irreducible MC, $A_x := \{n \ge 1 : P^n(x,x) > 0\}$ must be non-empty, thus $d_x = \text{g.c.d.}A_x$ is finite, and it further holds that all states have the same period $d \ge 1$ (*see* *the lecture for the proof*), called the period of the irreducible MC. If $d = 1$, the chain is said to be **aperiodic**.

• We can make connection between the **long-term behavior of** $P^n(x, y)$ and SD $\vec{\pi}$ in the following way (the proof was omitted in the lecture; please refer to the textbook). Consider an irreducible positive recurrent MC. We know such chain must have a SD, denoted by $\vec{\pi}$. Then, we have

(a) if the chain is aperiodic, then

$$\lim_{n\to\infty} P^n(x, y) = \pi(y), \tag{21}$$

for any $x, y \in S$.

(b) if the chain the periodic with period $d \geq 2$, then for any $x, y \in S$, there exists an integer

$$r \in \{0, 1, 2, \cdots, d-1\},$$

generally depending on x, y, such that

$$P^n(x, y) = 0 \tag{22}$$

for all n except that $n = md + r$ ($m \geq 0$ is an integer) for which

$$\lim_{m\to\infty} P^{md+r}(x, y) = d\pi(y). \tag{23}$$

This result tells that in case $d \geq 2$, we are able to determine the limits of subsequences

$$P^{md}, \quad P^{md+1}, \cdots, P^{md+(d-1)} \tag{24}$$

as $m \to \infty$. Precisely, for any given x, y,

$$P^{md}(x, y), \quad P^{md+1}(x, y), \cdots, P^{md+(d-1)}(x, y) \tag{25}$$

are zeros except that exactly one of them tends to $d\pi(y)$ as $m \to \infty$.

——**End, Updated on March 17**——