**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH4240 - Stochastic Processes - 2024/25 Term 2**

# Chapter I Markov Chain

## 1 Definition and Examples

• Let $S$ be a finite or countably infinite set of integers. For instance, $S = \{0, 1, \cdots, N\}$ ($N$ can be finite or infinite). Each element of $S$ is called a **state** and $S$ called the **state space**.

• Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of r.v. defined on a common probability space taking common values in $S$.

• $\{X_n\}_{n=0}^{\infty}$ is called a **Markov chain** (MC) if

$$P(X_{n+1} = x_{n+1}|X_0 = x_0, \cdots, X_n = x_n) = P(X_{n+1} = x_{n+1}|X_n = x_n) \tag{1}$$

for all states $x_0, \cdots, x_{n+1}$ in $S$. Identity (1) is called the **Markov property** meaning that given the present state, the past states have no influence on the future.

• For a MC $\{X_n\}_{n=0}^{\infty}$, $P(X_{n+1} = x_{n+1}|X_n = x_n)$ is called the **transition probability**, and if it is independent of $n$, we denote

$$P(x, y) = P(X_{n+1} = y|X_n = x) \quad (n = 0, 1, \cdots) \tag{2}$$

which is called the transition probability from state $x$ to state $y$. In such situation, the MC is **time homogeneous**. In the course, ONLY time homogeneous MCs will be discussed.

• It is clear to see

$$P(x, y) \geq 0; \quad \sum_{y \in S} P(x, y) = 1. \tag{3}$$

• A convenient way to represent the transition function $P(x, y)$ is to use the matrix form, i.e., for $S = \{0, 1, \cdots, N\}$,

$$P = [P(x, y)] = \begin{pmatrix} P(0,0) & P(0,1) & \cdots & P(0,N) \\ P(1,0) & P(1,1) & \cdots & P(1,N) \\ \cdots & \cdots & \cdots & \cdots \\ P(N,0) & P(N,1) & \cdots & P(N,N) \end{pmatrix}, \tag{4}$$

which is called the **transition matrix** (Markov matrix). Note that each row vector is a probability vector.

• Examples (See the textbook and course lectures):

(a) An i.i.d. chain.

(b) Two-state MC.

(c) Random walk.

(d) Gambler's ruin chain.

(e) Queuing chain.

(f) Branching chain.

# 2 Some Computational Issues

• Let $\{X_n\}_{n=0}^{\infty}$ be a time-homogeneous MC with the state $S = \{k\}_{k=0}^{N}$ ($N$ : finite or infinite) and the transition matrix $P$.

• **Question 1:** How to compute p.d.f. of $X_n$ ($n \geq 1$)?

♣ For $n = 0, 1, \cdots$, set

$$\Pi_n = [P(X_n = 0), P(X_n = 1), \cdots, P(X_n = N)], \tag{5}$$

denoting the p.d.f. of $X_n$ in a row-vector form. Noting

$$P(X_{n+1} = j) = \sum_i P(X_{n+1} = j | X_n = i) P(X_n = i) = \sum_i P(i,j) P(X_n = i), \tag{6}$$

one can show

$$\Pi_{n+1} = \Pi_n P, \tag{7}$$

namely, "the $j^{\text{th}}$ entry of $\Pi_{n+1}$" $= \Pi_n \times$ "the $j^{\text{th}}$ row of $P$". Further by induction,

$$\Pi_n = \Pi_0 P^n, \tag{8}$$

where $P^n$ is the $n$th power of the transition matrix $P$. By the rule of matrix product, the entry at the $x$th row and the $y$th column of $P^n$ is given by

$$P^n(x,y) = \sum_{x_1, x_2, \cdots, x_{n-1}} P(x, x_1) P(x_1, x_2) \cdots P(x_{n-1}, y), \tag{9}$$

where the sum is taken over all states $x_1, x_2, \cdots, x_{n-1}$ in $S$.

♣ In general it is not easy to directly compute $P^n$ for $n$ large. However, when $P$ can be diagonalisable in the sense that

$$P = QDQ^{-1}, \tag{10}$$

where $D = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N)$, then

$$P^n = QD^nQ^{-1} = Q\mathrm{diag}(\lambda_0^n, \lambda_1^n, \cdots, \lambda_N^n)Q^{-1}. \tag{11}$$

Using the above formula, it is also convenient to find the **limiting transition matrix** $\lim_{n\to\infty} P^n$ whenever all limits on the right-hand side of (11) exist.

♣ $P^n(x, y)$ can be used for computations in the following way:

$$P(X_n = y) = \sum_x \Pi_0(x)P^n(x, y), \tag{12}$$

$$P(X_n = y|X_0 = x) = P^n(x, y), \tag{13}$$

$$P(X_{m+n} = y|X_m = x) = P^n(x, y), \quad m = 0, 1, \cdots . \tag{14}$$

These motivate one to introduce a new meaning of the matrix $P^n(x, y)$ $(n = 0, 1, 2, \cdots)$, which is called the **$n$-step transition function**, giving the probability that the chain starting at $x$ visits $y$ in $n$ steps. Correspondingly, $P^n$ is called the **$n$-step transition matrix**. We set $P^0 = I$, i.e.,

$$P^0(x, y) = \delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

• **Question 2:** How to compute

$$P(X_n = y \text{ for some } n \geq 1|X_0 = x), \tag{16}$$

that is the conditional probability that the chain starting at $x$ ever visits $y$ in finite time? Denote this probability by $\rho_{xy}$.

♣ Let $A \subset S$. The **hitting time** $T_A$ of $A$ is defined by

$$T_A = \min\{n \geq 1 : \ X_n \in A\}. \tag{17}$$

Thus, $T_A$ is also a r.v. denoting the first positive time the chain hits $A$, with

$$\mathrm{Rang}\, T_A = \{1, 2, 3, \cdots\} \cup \{\infty\}, \tag{18}$$

and we set $T_A = \infty$ if $X_n \notin A$ for all $n \geq 1$. Note that for $m = 1, 2, \cdots$

$$\{T_A = m\} = \{X_1 \notin A, X_2 \notin A, \cdots, X_{m-1} \notin A, X_m \in A\}. \tag{19}$$

When $A = \{y\}$, we write

$$T_y := T_{\{y\}} = \min\{n \geq 1 : \ X_n = y\}, \tag{20}$$

meaning the first positive time the chain visits $y$. It is clear to see

$$\rho_{xy} = P_x(T_y < \infty) = \sum_{k=1}^{\infty} P_x(T_y = k), \tag{21}$$

as the event $\{T_y < \infty\}$ exactly means $\{X_n = y, \text{ for some } n \geq 1\}$, i.e., the chain starting at $x$ visits $y$ at some positive time. Also,

$$1 - \rho_{xy} = P_x(T_y = \infty). \tag{22}$$

♣ One has

$$P_x(T_y = 1) = P(x, y), \tag{23}$$
$$P_x(T_y = n + 1) = \sum_{z \neq y} P(x, z) P_z(T_y = n), \quad n \geq 1, \tag{24}$$
$$P^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y, y). \tag{25}$$

Basically, they are, respectively, due to the fact that for a chain starting at $x$,

$$\{T_y = 1\} = \{X_1 = y\}, \tag{26}$$
$$\{T_y = n + 1\} = \cup_{z \neq y} \{X_1 = z, X_2 \neq y, \cdots, X_n \neq y, X_{n+1} = y\}, \quad n \geq 1, \tag{27}$$
$$\{X_n = y\} = \cup_{m=1}^{n} \{T_y = m, X_n = y\}. \tag{28}$$

In (25), if $y$ is an absorbing state, then $P^k(y, y) = 1$ for any $k \geq 0$. Thus (25) reduces to

$$P^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m) = P_x(T_y \leq n), \tag{29}$$

for an absorbing state $y$.

♣ To compute the matrix $[\rho_{xy}]$ from the transition matrix $[P(x, y)]$, one may use the following formula

$$\rho_{xy} = P(x, y) + \sum_{z \neq y} P(x, z) \rho_{zy}. \tag{30}$$

This is due to that for a chain starting at $x$,

$$\{T_y < \infty\} = \{X_1 = y\} \cup \{X_1 \neq y, X_n = y \text{ for some } n \neq 2\} \tag{31}$$
$$= \{X_1 = y\} \cup (\cup_{y \neq z \in S} \{X_1 = z, X_n = y \text{ for some } n \neq 2\}), \tag{32}$$

with corresponding probabilities computed as

$$P_x(X_1 = y) = P(x, y), \tag{33}$$

and for $z \neq y$,

$$P_x(\{X_1 = z, X_n = y \text{ for some } n \neq 2\}) \tag{34}$$
$$= P_x(X_1 = z)P_x(X_n = y \text{ for some } n \neq 2 | X_1 = z) \tag{35}$$
$$= P(x, z)\rho_{zy}. \tag{36}$$

Let $S = \{1, 2, \cdots, N\}$, for instance. Then, fixing $j \in S$, the formula (30) means

$$\begin{pmatrix} \rho_{1j} \\ \rho_{2j} \\ \vdots \\ \rho_{Nj} \end{pmatrix} = \begin{pmatrix} P(1,j) \\ P(2,j) \\ \vdots \\ P(N,j) \end{pmatrix} + \begin{pmatrix} P(1,1) & \cdots & P(1,j-1) & 0 & P(1,j+1) & \cdots & P(1,N) \\ P(2,1) & \cdots & P(2,j-1) & 0 & P(2,j+1) & \cdots & P(2,N) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ P(N,1) & \cdots & P(N,j-1) & 0 & P(N,j+1) & \cdots & P(N,N) \end{pmatrix} \begin{pmatrix} \rho_{1j} \\ \rho_{2j} \\ \vdots \\ \rho_{Nj} \end{pmatrix}, \tag{37}$$

where the first term on the right is just the $j$th column of $P$, and the coefficient matrix of the second term on the right is just $P$ with the $j$th column replaced by zeros.

*Warning:* At the present time, it is unclear that this linear system of equations is solvable, i.e., either there is a unique solution, or there is no solution, or there are infinite number of solutions.

• **Question 3:** **Times of visit to a state**. For a chain starting at $x \in S$, we denote $N(y)$ to be the NO of times that $X_n$ $(n \geq 1)$ visits $y$. Note that $N(y)$ is a r.v. taking values in

$$\text{Range } N(y) = \{0, 1, 2, \cdots\} \cup \{\infty\}. \tag{38}$$

For $k = 0$, the event $\{N(y) = 0\}$ means that $y$ is not visited at any positive time. For $k = 1, 2, \cdots$, $N(y) = k$ means that $y$ is visited exactly $k$ times. For $k = \infty$, $\{N(y) = \infty\}$ means that $y$ is visited infinitely times. Here and below, when we mention the event $\{N(y) \geq k\}$, it precisely means

$$\{k \leq N(y) \leq \infty\} = \{k \leq N(y) < \infty\} \cup \{N(y) = \infty\}. \tag{39}$$

♣ We may evaluate the pdf of $N(y)$ in the following way.

$$P_x(N(y) \geq 1) = P_x(T_y < \infty) = \rho_{xy}, \tag{40}$$
$$P_x(N(y) = 0) = 1 - P_x(N(y) \geq 1) = 1 - \rho_{xy}. \tag{41}$$

For $k = 1, 2, \cdots$,

$$P_x(N(y) \geq k) = \rho_{xy}\rho_{yy}^{k-1}, \tag{42}$$
$$P_x(N(y) = k) = P_x(N(y) \geq k) - P_x(N(y) \geq k+1) = \rho_{xy}\rho_{yy}^{k-1}(1 - \rho_{yy}), \tag{43}$$

where $1 - \rho_{yy}$ gives the probability that the chain initially from $y$ never visits $y$ again in any finite time. And, for $k = \infty$,

$$P_x(N(y) = \infty) = \lim_{k\to\infty} P_x(N(y) \geq k) \tag{44}$$

$$= \lim_{k\to\infty} \rho_{xy}\rho_{yy}^{k-1} \tag{45}$$

$$= \begin{cases} 0 & \text{if } \rho_{yy} < 1, \textit{ i.e.}, y \text{ is transient;} \\ \rho_{xy} & \text{if } \rho_{yy} = 1, \textit{ i.e.}, y \text{ is recurrent.} \end{cases} \tag{46}$$

♣ Moreover, we would consider the expectation of $N(y)$, denoted by $E_x(N(y))$ meaning the **expected NO of visit** to $y$ from $x$. It is clear to see that if $P_x(N(y) = \infty)$ is positive then $E_x(N(y)) = \infty$. In general, one has

(a) $y$ is transient iff $P_y(N(y) = \infty) = 0$, iff $E_y(N(y)) < \infty$. For a transient state $y$,

$$E_x(N(y)) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty, \quad x \in S. \tag{47}$$

(b) $y$ is recurrent iff $P_y(N(y) = \infty) = 1$, iff $E_y(N(y)) = \infty$.

These results are quite obvious heuristically. For instance, $y$ is recurrent if and only if the chain starting at $y$ must visit $y$ at some positive time, if and only if the chain starting at $y$ must visit $y$ at infinitely number of times, if and only if the expected NO of times of visit to $y$ from $x$ is infinite. $y$ is transient if and only if $y$ is visited at only finite number of times, if and only if it is IMPOSSIBLE that the chain visits $y$ at infinitely number of times. The formula (47) is a consequence of (43) as well as $P_y(N(y) = \infty) = 0$, due to

$$E_x(N(y)) = \sum_{k=1}^{\infty} kP_x(N(y) = k). \tag{48}$$

There is another way to compute $E_x(N(y))$ which could be infinite or finite. Defining

$$1_y(X_n) = \begin{cases} 1 & \text{if } X_n = y, \\ 0 & \text{otherwise,} \end{cases} \tag{49}$$

we see

$$N(y) = \sum_{n=1}^{\infty} 1_y(X_n). \tag{50}$$

Then,

$$E_x(N(y)) = E_x\left(\sum_{n=1}^{\infty} 1_y(X_n)\right) = \sum_{n=1}^{\infty} E_x(1_y(X_n)) = \sum_{n=1}^{\infty} P^n(x,y). \tag{51}$$

From this formula, we can claim that *a finite state space must contain at least one recurrent state.* Refer to the lecture note for the rigorous proof; a heuristic argument is based on the fact that if all states in a finite state space are transient, i.e., all states are visited at only finite number of times, then the total NO of times of visit to all states must be finite, which obviously contradicts the fact that time of the MC can go to infinity!

• **Question 4:** Decomposition of state space.

♣ $x$ **leads to** $y$ ($x \to y$) if $\rho_{xy} > 0$. ($x \to y$ means that if the chain starts at $x$ then the chain will visit $y$ with a positive probability $\rho_{xy}$.)

♣ Note the following two facts:

(a) $x \to y$ if and only if $P^n(x, y) > 0$ for some $n \geq 1$.

(b) If $x \to y$ and $y \to z$ then $x \to z$.

♣ Let $x$ be recurrent and $x \to y$. Then,

(a) $y \to x$.

(b) $y$ is recurrent also.

(c) $\rho_{xy} = \rho_{yx} = 1$.

Instead of giving the rigorous proof of the above argument (see the textbook Pages 21–22), we better understand it in a heuristic way. (a): As $\rho_{xx} = 1$ and $\rho_{xy} > 0$, it is necessarily $\rho_{yx} > 0$. Otherwise $\rho_{yx} = 0$, then $x$ cannot be recurrent! (*why?*) (b): Think about the fact that the chain starting from $y$ can visit $x$ ($\rho_{yx} > 0$), then must re-visit $x$ at infinitely number of times ($P_x(N(x) = \infty) = 1$, as $x$ is recurrent), and finally can visit from $x$ to $y$ ($\rho_{xy} > 0$). This implies that the chain starting at $y$ MUST re-visit $y$ at infinitely number of times, i.e., $P_y(N(y) = \infty) = 1$, so $y$ is recurrent. (Otherwise, $y$ is visited only at finite number of times, so $x$ is also visited at finite number of times, a contradiction!) (c): It suffices to argue $\rho_{yx} = 1$, as you can interchange $x$ and $y$ to get $\rho_{xy} = 1$. To see $\rho_{yx} = 1$, otherwise $\rho_{yx} < 1$. It means that with the positive probability $1 - \rho_{yx}$ the chain starting at $y$ will never visit $x$. This obviously contradicts the fact that $x$ is recurrent because the chain staring at $x$ can visit $y$ ($\rho_{xy} > 0$) and then will never visit back to $x$ with the positive probability $1 - \rho_{yx}$.

♣ Definitions:

(i) $C \subset S$ is **closed** if $\rho_{xy} = 0$, $\forall\, x \in C$, $\forall\, y \notin C$, i.e., no state in $C$ leads to any state out $C$, equivalently to say that, if $x \to y$ for $x \in C$ and $y \in S$, then $y \in C$.

(ii) A closed set $C \subset S$ is **irreducible** if $x \to y$, $\forall\, x \in C$, $\forall, y \in C$, i.e., any state in $C$ leads either to itself or to any other state in $C$. (i.e., all states in $C$ can communicate with each other!)

(iii) A MC is irreducible if its state space $S$ is irreducible.

Directly in terms of the transition matrix $P$, it is easy to justify whether or not a set $C$ is closed. In fact, one can show that $C$ is closed if and only if

$$P(x, y) = 0, \ \forall\, x \in C, \forall\, y \notin C. \tag{52}$$

Note that if $C$ is closed, $x \in C$, and $P(x, y) > 0$, then $y$ must be in $C$. Note also that if $C \subset S$ is closed then the MC can also be regarded as a MC with the state space $C$.

♣ Defining $S_R$ to be a set of all recurrent states and $S_T$ to be a set of all transient states, then $S_R \cap S_T = \emptyset$ and

$$S = S_R \cup S_T. \tag{53}$$

Note that any state in $S_R$ cannot lead to a state in $S_T$. So, $S_R$ is a closed set of all recurrent states.

Assume $S_R \neq \emptyset$, for instance, there is a recurrent state $x_0 \in S_R$. Define

$$C_{x_0} = \{x \in S_R : \ x_0 \to x\}. \tag{54}$$

Then, one can show that $C_{x_0}$ must be closed and irreducible (see the lecture note). You see that $C_{x_0}$ is the *largest* closed irreducible set of recurrent states that contains $x_0$ (show it!)

Moreover, one can also show that if $C_1$ and $C_2$ are two irreducible and closed sets then either $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$ (see the lecture note). Therefore, one can conclude that if $S_R \neq \emptyset$ then

$$S_R = \cup_{i=1}^{k} C_i, \tag{55}$$

for some finite or infinite $k \geq 1$, where $C_i$, $1 \leq i \leq k$, are disjoint irreducible closed sets of recurrent states.

Note from the above that if $C$ is an irreducible and closed set, then either $C \subset S_R$ (i.e., all of states in $C$ are recurrent) or $C \subset S_T$ (i.e., all states in $C$ are transient). In particular, if $C$ is a finite irreducible closed set, then $C \subset S_R$. (*Why?*)

In terms of the **decomposition of state space**

$$S = S_R \cup S_T = \left(\cup_{i=1}^{k} C_i\right) \cup S_T, \tag{56}$$

we may rewrite $P$ as its **canonical form**:

$$\widetilde{P} = \begin{array}{c||c|c|c|c|c}
 & \boxed{C_1} & \boxed{C_2} & \cdots & \boxed{C_k} & \boxed{S_T} \\ \hline
\boxed{C_1} & \boxed{*} & 0 & \cdots & 0 & 0 \\ \hline
\boxed{C_2} & 0 & \boxed{*} & \ddots & 0 & 0 \\ \hline
\vdots & \vdots & \ddots & \boxed{*} & 0 & 0 \\ \hline
\boxed{C_k} & 0 & 0 & 0 & \boxed{*} & 0 \\ \hline
\boxed{S_T} & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*} & \boxed{*}
\end{array}, \tag{57}$$

where $\boxed{*}$ denotes the matrix with possible nonzero entries.

♣ Absorption probability: Let $C$ be an irreducible and closed set of recurrent states. Let $T_C$ be the hitting time of $C$, i.e., the first positive time the chain enters $C$. Consider the function

$$\rho_C(x) := P_x(T_C < \infty), \tag{58}$$

that is the probability that the chain from $x \in S$ enters $C$ in finite time. Note that once the chain hits $C$, it remains in $C$ forever. Thus $\rho_C(\cdot)$ is usually called the **absorption probability**. It is clear to see $\rho_C(x) = 1$ if $x \in C$, and $\rho_C(x) = 0$ if $x$ is not in $C$ but still recurrent. Therefore, the general situation is to compute $\rho_C(x)$ for $x \in S_T$, i.e. $x$ is transient.

In fact, one can check (Exercise) that

$$\rho_C(x) = \sum_{y\in C} P(x,y) + \sum_{y\in S_T} P(x,y)\rho_C(y), \quad x \in S_T. \tag{59}$$

This formula is quite obvious heuristically: the event the chain from $x \in S_T$ enters $C$ *in finite time* is equal to the disjoint union of the following events for this chain from $x$:

— the chain visits by one step each possible state $y$ in $C$.
— the chain first visits by one step each possible state $y$ in $S_T$ and then enters $C$ in finite time.

One can further rigorously show (see the textbook; the complete proof was ignored in lecture) that *if $S_T$ is finite then* (59) *admits a unique solution $\rho_C(x)$, $x \in S_T$.* An alternative proof can be given as follows. Suppose $S$ has the state decomposition $S = C_1 \cup C_2 \cup S_T$, for instance. After reordering $S$, we assume to have the canonical transition matrix

$$P = \begin{array}{c} \\ C_1 \\ C_2 \\ S_T \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & S_T \end{array} \\ \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ S_1 & S_2 & Q \end{bmatrix} \end{array}. \tag{60}$$

To compute $\rho_{C_1}(x)$, $x \in S_T$, we let $S_T = \{x_1, \cdots, x_\ell\}$ and denote the column vectors

$$\vec{v} := \begin{bmatrix} \rho_{C_1}(x_1) \\ \vdots \\ \rho_{C_1}(x_\ell) \end{bmatrix}, \quad \vec{v}_0 := \begin{bmatrix} \sum_{y\in C_1} P(x_1,y) \\ \vdots \\ \sum_{y\in C_1} P(x_\ell,y) \end{bmatrix}, \tag{61}$$

so (59) can be written as the matrix form

$$\vec{v} = \vec{v}_0 + Q\vec{v}, \tag{62}$$

which has a unique solution $\vec{v} = (I-Q)^{-1}\vec{v}_0$, because $Q$ has all eigenvalues with moduli strictly less than one (*why?* see the tutorial) and thus $I - Q$ is invertible. Note that in case $C_1$ contains a single absorbing state, then $\vec{v}_0 = S_1$ and $\vec{v} = (I-Q)^{-1}S_1$, and we also will come back to this formula in the next Chapter.

It is also obvious to see that if $S_T$ is finite, then

(i)
$$\sum_i \rho_{C_i}(x) = 1, \quad x \in S_T, \tag{63}$$

where $C_i$ correspond to those in the decomposition (56). It means that the chain from $x \in S_T$ enters $S_R$ in finite time for sure! ($\because S_T$ is finite and each transient state is visited only finite times, hence $S_T$ is visited only finite times, and afterwards the chain enters $S_R$ for sure!)

(ii)
$$\rho_{xy} = \rho_{C_i}(x), \quad x \in S_T,\ y \in C_i. \tag{64}$$

It means that for a chain from $x \in S_T$, one has the same probabilities that the chain visits $y$ in $C_i$ in finite time and that the chain enters $C_i$ in finite time. (Note: Given $X_0 = x \in S_T$, two events $\{T_y < \infty\}$ and $\{T_{C_i} > \infty\}$ are the same provided that $y \in C_i$)

# 3 More Examples

- **Birth and Death Chain:**

(a) $\{X_n\}_{n=0}^{\infty}$, $S = \{0, 1, 2, \cdots, d\}$ ($d$ is finite or $\infty$). When $d$ is finite,

$$P = \begin{pmatrix} r_0 & p_0 & & & & \\ q_1 & r_1 & p_1 & & & \\ & q_2 & r_2 & p_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & q_{d-1} & r_{d-1} & p_{d-1} \\ & & & & q_d & r_d \end{pmatrix}. \tag{65}$$

When $d = \infty$,

$$P = \begin{pmatrix} r_0 & p_0 & & & \\ q_1 & r_1 & p_1 & & \\ & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{66}$$

(b) For the chain starting at some state $x \in (a, b)$ with $a < b$, we are interested in computing

$$P_x(T_a < T_b) \quad \text{or} \quad P_x(T_a > T_b). \tag{67}$$

See the lecture for the general approach; you have to be familiar with the full procedure of such computations. Note that the above expressions seem not well-defined at $x = a$ and $b$. However, in such trivial cases, it is still reasonable to assign values in the way that

$$P_a(T_a < T_b) = 1, \quad P_b(T_a < T_b) = 0. \tag{68}$$

Here, $T_a < T_b$ is understood to be an event $A$ that the chain $\{X_n\}_{n=0}^{\infty}$ is at $a$ strictly earlier than $b$. Then, if $X_0 = a$ then $A$ occurs for sure, i.e. $P_a(A) = 1$; if $X_0 = b$ then $A$ does not occur for sure, i.e. $P_b(A) = 0$.

(c) Note that the chain is irreducible if and only if $p_x > 0$, $\forall\, x \geq 0$ and $q_x > 0$, $\forall\, x \geq 1$. In such situation, we see that if $d$ is finite then the chain is recurrent, i.e., all states in $S$ are recurrent. However, if $d = \infty$, it is NOT obvious to see whether such irreducible chain is recurrent or transient!!! In fact, we can derive a criterion to justify it; see the lecture.

- **Branching Chain:**

(a) Each particle generates $\xi$ (r.v.) particles independently in the next generation, and $X_n$ denotes the total NO of the $n$-th generation. Recall that $S = \{0, 1, 2, \cdots\}$, $P(0, 0) = 0$, and

$$P(x, y) = P(\xi_1 + \xi_2 + \cdots + \xi_x = y), \quad x \geq 1. \tag{69}$$

What we need to understand from this MC $\{X_n\}_{n=0}^{\infty}$ is to determine

$$\rho := \rho_{10} = P_1(T_0 < \infty), \tag{70}$$

i.e., the probability that the descendants of a given particle eventually become extinct. Thus, $\rho$ is called the **extinction probability** of the chain. We have two trivial situations $p_0 = 0$ and $p_0 = 1$ corresponding to which $\rho = 0$ and $\rho = 1$, respectively. WLG, we assume $0 < p_0 < 1$.

(b) Assume that $\xi$ has the p.d.f.

$$p_k := P(\xi = k), \quad k \geq 0, \tag{71}$$

and the mean

$$\mu := E(\xi) = \sum_{k=0}^{\infty} k p_k. \tag{72}$$

One can show that $\rho$ solves

$$\rho = \sum_{k=0}^{\infty} p_k \rho^k. \tag{73}$$

The existence and uniqueness of solutions to $t = \Phi(t)$ can be clarified as follows; here,

$$\Phi(t) = \sum_{k=0}^{\infty} p_k t^k, \tag{74}$$

which is called the **moment generating function** of the p.d.f. of r.v. $\xi$.

(i) If $\mu < 1$, $\exists!\, \rho = 1$ (extinct for sure!)

(ii) If $\mu = 1$, $\exists!\, \rho = 1$ (extinct for sure!)

(iii) If $\mu > 1$, $\exists!\, \rho \in (0,1)$ (extinct with the probability $\rho \in (0,1)$)

(c) For $\mu < 1$, one can also directly show that

$$P_1(T_0 > n) \le E(X_n) = \mu^n E(X_0) \to 0 \quad \text{as } n \to \infty, \tag{75}$$

$$\rho_{10} = P_1(T_0 < \infty) = 1 - \lim_{n\to\infty} P_1(T_0 > n) = 1. \tag{76}$$

See the lecture for details.

- **Queuing Chain:**

(a) $\xi_n$ denotes the NO of arrivals in the $n$-th unit time. $\{\xi_n\}_{n=1}^{\infty}$ are i.i.d.r.v. with p.d.f.

$$f(k) = p_k, \quad k = 0, 1, 2, \cdots. \tag{77}$$

Exactly one and only one customer is served and leaves the waiting line at the end of a unit time if there is at least one person on the line at the beginning of the unit time. $X_n$ denotes the NO of customers in the waiting line. Then, $S = \{0, 1, 2, \cdots\}$, and

$$P(0, y) = f(y), \quad y \ge 0, \tag{78}$$

$$P(x, y) = f(y - (x-1)), \quad x \ge 1, y \ge x - 1. \tag{79}$$

Note $P(0, y) = P(1, y)$, $y \ge 0$.

(b) The first question is when the chain is irreducible. See Exercises Q37 on page 46 in the textbook. For instance, if $p_0 > 0$ and $p_0 + p_1 < 1$ then the chain must be irreducible.

(c) The second question is that, assuming that the chain is irreducible, we are interested in deciding if the chain is recurrent or transient, for instance, letting

$$\rho := \rho_{00} = P_0(T_0 < \infty), \tag{80}$$

we want to decide $\rho = 1$ or $\rho < 1$. Once again one can show that $\rho$ solves

$$t = \Phi(t), \tag{81}$$

where $\Phi(t) = \sum_{k=0}^{\infty} p_k t^k$ is the moment generating function of $(p_k)_{k\geq 0}$. See the lecture for the proof, for instance, it is based on

$$\rho_{10} = P(1,0) + \sum_{k=1}^{\infty} P(1,k)\rho_{k0}, \tag{82}$$

together with $\rho_{10} = \rho_{00} := \rho$ and $\rho_{k0} = \rho_{k,k-1}\rho_{10} = \rho_{10}^k = \rho^k$.

(d) Again, the existence of solutions to $t = \Phi(t)$ can be assured in terms of $\mu := E(\xi)$ in the same way as in the Branching Chain.

——**End, updated on Feb 13**——