Computation of the expected time from x to state y[1] [2]

We consider an irreducible Markov jump process $\{X_t\}_{t\geq 0}$ with state space S and rate matrix D. Let a state $y \in S$ be fixed and

$$T_y = \inf\{t > 0 : X_t = y\}.$$

We are going to compute

$$\boxed{w(x) := E(T_y|X_0 = x) = E_x(T_y),} \tag{1}$$

for any $x \in S$, that's the *mean passage time* to y for X_t starting at x.

For $x = y$, it is obvious to see $T_y = 0$ given $X_0 = y$, so $E_y(T_y) = 0$, i.e., $w(y) = 0$.

For $x \neq y$, let $X_0 = x$ and let τ_x be the waiting time for the first jump. Then,

$$\begin{aligned} E_x(T_y) &= E(\tau_x|X_0 = x) \\ &\quad + \sum_{\substack{z\in S\\ z\neq x,y}} P(X_{\tau_x} = z|X_0 = x)E(T_y|X_0 = z). \end{aligned} \tag{2}$$

Since τ_x is exponential with parameter q_x,

$$\begin{cases} E(\tau_x|X_0 = x) = \dfrac{1}{q_x}, \\ P(X_{\tau_x} = z|X_0 = x) = Q_{xz} = \dfrac{q_{xz}}{q_x}. \end{cases}$$

With this, (2) is reduced to

$$w(x) = \frac{1}{q_x} + \sum_{z\neq x,y} \frac{q_{xz}}{q_x} w(z).$$

Further multiplying it by q_x,

$$q_x w(x) = 1 + \sum_{z\neq x,y} q_{xz} w(z), \quad y \neq x \in S. \tag{3}$$

We give an example to explain how to solve the above linear system for unknowns $w(x)$ with all $x \in S$ except $x = y$. For instance, let $S = \{1, 2, 3, 4\}$ and $y = 3$. Taking $x = 1, 2, 4$ in (3) gives

$$q_1 w(1) = 1 + \sum_{z\neq 1,3} q_{1z} w(z),$$
$$q_2 w(2) = 1 + \sum_{z\neq 2,3} q_{2z} w(z),$$
$$q_4 w(4) = 1 + \sum_{z\neq 4,3} q_{4z} w(z),$$

[1] This is an extra note that will not be tested in the final examination but it is good to learn.

[2] If you have any question to this note, please freely address it to the course instructor Renjun Duan at `rjduan@math.cuhk.edu.hk`.

or in matrix form,

$$\begin{bmatrix} q_1 & & \\ & q_2 & \\ & & q_3 \end{bmatrix} \begin{bmatrix} w(1) \\ w(2) \\ w(4) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & q_{12} & q_{14} \\ q_{21} & 0 & q_{24} \\ q_{41} & q_{42} & 0 \end{bmatrix} \begin{bmatrix} w(1) \\ w(2) \\ w(4) \end{bmatrix}.$$

That is,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \tilde{D} \begin{bmatrix} w(1) \\ w(2) \\ w(4) \end{bmatrix}, \quad \tilde{D} := \begin{bmatrix} -q_1 & q_{12} & q_{14} \\ q_{21} & -q_2 & q_{24} \\ q_{41} & q_{42} & -q_4 \end{bmatrix},$$

where the coefficient matrix $\tilde{D}$ can be obtained from the rate matrix

$$D = \begin{bmatrix} -q_1 & q_{12} & q_{13} & q_{14} \\ q_{21} & -q_2 & q_{23} & q_{24} \\ q_{31} & q_{32} & -q_3 & q_{34} \\ q_{41} & q_{42} & q_{43} & -q_4 \end{bmatrix}$$

by deleting the row and the column both associated to state $y = 3$.

Generally, for a fixed state $y \in S$, to solve $w(x) = E_x(T_y)$ with $x \neq y$ in terms of (3), we are reduced to solve

$$\vec{0} = \vec{1} + \tilde{D}\tilde{w}$$

as

$$\tilde{w} = -\tilde{D}^{-1}\vec{1}, \tag{4}$$

where the coefficient matrix $\tilde{D}$ can be obtained from the rate matrix $D = [q_{xy}]$ by deleting the row and the column both associated to state y. Notice that the diagonal entries of $\tilde{D}$ are all non-positive and at least one of its row sums is strictly negative. Hence we can conclude that all eigenvalues of $\tilde{D}$ have the negative real part and hence $\tilde{D}$ turns out to be invertible.

—END—