Solution 3

1. (1) For any $x, y \in \bigcap_{i\in I} X_i$, $\lambda \in [0,1]$
Then $x, y \in X_i$, $\forall i \in I$.
Since X_i, $i \in I$ are nonempty convex subsets.
Then $\lambda x + (1-\lambda)y \in X_i$, $\forall i \in I$
Thus, $\lambda x + (1-\lambda)y \in \bigcap_{i\in I} X_i$
Hence, $\bigcap_{i\in I} X_i$ is convex.
(2) For any $x, y \in \lambda_1 X_1 + \cdots + \lambda_k X_k$, $\lambda \in [0,1]$.
There exists $x_i, y_i \in X_i$, $1 \le i \le k$ such that

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k, \quad y = \lambda_1 y_1 + \cdots + \lambda_k y_k.$$

Since $X_1, \ldots, X_k \subseteq \mathbb{R}^n$ are nonempty convex subsets.
Then $x_i + (1-\lambda)y_i \in X_i$, $\forall 1 \le i \le k$.
Note that

$$\begin{aligned}\lambda x + (1-\lambda)y &= \lambda(\lambda_1 x_1 + \cdots + \lambda_k x_k) + (1-\lambda)(\lambda_1 y_1 + \cdots + \lambda_k y_k)\\ &= \lambda(\lambda_1(x_1 + (1-\lambda)y_1) + \cdots + \lambda_k(x_k + (1-\lambda)y_k))\end{aligned}$$

Since $x_i + (1-\lambda)y_i \in X_i$, $\forall 1 \le i \le k$. Then $\lambda x + (1-\lambda)y \in \lambda_1 X_1 + \cdots + \lambda_k X_k$.

2. Suppose K is a convex set. Denote the interior and closure by K^* and $\bar{K}$.

(1) For any $x, y \in K^*$ and $\lambda \in [0,1]$. There exists $r \in \mathbb{R}$ such that $B_r(x), B_r(y) \subseteq K$. Let $r' < r$. For any $z \in B_{r'}(x + (1-\lambda)y)$, we have $\lambda z + (1-\lambda)B_r(y)$. Thus, $\|x + (1-\lambda)y\| < r' < \lambda r$. Let $w = z - (\lambda x + (1-\lambda)y)$. Then $\|w\| < r'$. Hence, $z = w + \lambda x + (1-\lambda)y = \lambda(x+w) + (1-\lambda)(y+w)$. Note that $\|x + w - x\| = \|w\| < r$, $\|y + w - y\| = \|w\| < r$. Then $x + w \in B_r(x)$ and $y + w \in B_r(y)$. Thus, $x + w \in K$ and $y + w \in K$. Since K is convex, then $\lambda(x+w) + (1-\lambda)(y+w) \in K$. Thus, $z \in K$. Therefore, $B_{r'}(x + (1-\lambda)y) \subseteq K$. Hence, $\lambda x + (1-\lambda)y \in K^*$.

(2) For any $x, y \in \bar{K}$ and $\lambda \in [0,1]$. Then there exists $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq K$ such that $x_n \to x$, $y_n \to y$. Since K is convex. Then $\lambda x_n + (1-\lambda)y_n \in K$, $\forall k = 1, 2, \ldots$. Note that $\lim_{n\to\infty} \lambda x_n + (1-\lambda)y_n = \lambda \lim_{n\to\infty} x_n + (1-\lambda)\lim_{n\to\infty} y_n = \lambda x + (1-\lambda)y$. Thus, $\lambda x + (1-\lambda)y \in \bar{K}$.

3.Let $T : V \to W$ be a linear transformation.

(1) Suppose V is a convex set and $W = T(V)$.

For any $x, y \in W$, $\lambda \in [0, 1]$. There exist $u, v \in V$ such that $x = T(u)$, $y = T(v)$.

Note that $x + (1 - \lambda)y = \lambda T(u) + (1 - \lambda)T(v) = T(\lambda u) + T((1 - \lambda)v) = T(\lambda u + (1 - \lambda)v)$.

Since V is convex. Then $\lambda u + (1 - \lambda)v \in V$.

Thus, $x + (1 - \lambda)y = T(\lambda u + (1 - \lambda)v) \in W$.

Hence, W is convex.

(2) Suppose W is convex. Consider the set $T^{-1}(W) = \{x \in V : T(x) \in W\}$.

For any $u, v \in T^{-1}(W)$ and $\lambda \in [0, 1]$. Note that $T(u), T(v) \in W$.

Since W is convex.

Then $\lambda T(u) + (1 - \lambda)T(v) = T(\lambda u + (1 - \lambda)v) \in W$.

Thus, $\lambda u + (1 - \lambda)v \in T^{-1}(W)$.

Hence, $T^{-1}(W)$ is convex.