Solution 2

1. (1) Since A is symmetric, it can be orthogonally diagonalized. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal set of eigenvectors of A with corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For any unit vector $x \in \mathbb{R}^n$, express x as $x = \sum_{i=1}^n \alpha_i u_i$ where $\sum_{i=1}^n \alpha_i^2 = 1$. Then:

The minimum value of this expression occurs when all weight is assigned to the smallest eigenvalue λ_1, i.e., $\alpha_1 = 1$ and $\alpha_i = 0$ for $i \geq 2$. Thus,

$$m = \lambda_1$$

which is the smallest eigenvalue of A.

(2) Extend $\{v_1, \ldots, v_k\}$ to an orthonormal eigenbasis $\{v_1, \ldots, v_n\}$ of A with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The constraint $v_i^T x = 0$ for $i = 1, \ldots, k$ restricts x to the subspace $S = \text{span}\{v_{k+1}, \ldots, v_n\}$. Any $x \in S$ with $\|x\| = 1$ can be written as $x = \sum_{i=k+1}^n \beta_i v_i$ where $\sum_{i=k+1}^n \beta_i^2 = 1$. Then:

The minimum is achieved when $\beta_{k+1} = 1$ and $\beta_i = 0$ for $i > k+1$, yielding $x = v_{k+1}$. Thus,

$$m = \lambda_{k+1}$$

which is an eigenvalue of A.

Remark: Both results follow from the Rayleigh-Ritz theorem and the Courant-Fischer minimax principle for symmetric matrices.

2. Recall the projection formula

$$x = y + \frac{-c^T y + d}{\|c\|} c$$

Necessary condition

$$x_i - y_i + \lambda c_i = 0$$

$$x_i = y_i - \lambda c_i$$

or

$$\sum c_i x_i = d$$

hence

$$d = \sum c_i y_i - \lambda \sum c_i^2$$

$$\lambda = \frac{c^T y - d}{\|c\|}$$

3. Qualification The constraints are linear, hence qualified at every point.
Necessary Condition
(1)

$$\begin{cases} 2x - \lambda_1 - \lambda_2 & = 0 \\ 2y - \lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ 4z - \lambda_2 + \lambda_3 & = 0 \end{cases}$$

Saturation
Case $y = z\lambda_3$

$$\begin{cases} 2x - \lambda_1 - \lambda_2 & = 0 \\ 2y - \lambda_1 - 2\lambda_2 + \lambda_3 & = 0 \\ \lambda_3 = -\lambda_2 + 4z \end{cases}$$

We obtain the system

$$\begin{cases} 2x - \lambda_1 - \lambda_2 & = 0 \\ 6y - \lambda_1 - 3\lambda_2 & = 0 \end{cases}$$

Case $x + y = 1 \quad \lambda_1$

$$x = 1 - y$$

We find that the function to be minimized is

$$f = 1 - 2y + y^2 + y^2 + 2y^2 = 4\left(y - \frac{1}{4}\right)^2 + \frac{3}{4}$$

with the constraint $1 + 2y \geq 0$ We obtain a point that verifies the necessary condition at

$$x = \frac{3}{4},\ y = \frac{1}{4},\ z = \frac{1}{4}$$

Sub-case $x + y > 1 \quad \lambda_1 = 0$
We obtain $x = y = z$ We get $y > \frac{1}{2}$ Then $2y = \lambda_2 > 0$ hence $4y = 0$ impossible.
Case $y < z \quad \lambda_3 = 0$

$$\begin{cases} 2x - \lambda_1 - \lambda_2 & = 0 \\ 2y - \lambda_1 - 2\lambda_2 & = 0 \\ 4z - \lambda_2 & = 0 \end{cases}$$

Sub-case $x+y=1$ We then have $\lambda_1 = 1-6z$ and thus $2y = 1+2z$ impossible since $y<z$

Sub-case $x+y>1 \quad \lambda_1 = 0$

$x=2z$ and $y=4z$ with $z>0$ impossible since $y<z$

Conclusion

Minimum of $\frac{3}{4}$ at $x=\frac{3}{4}$, $y=\frac{1}{4}$, $z=\frac{1}{4}$

(2)

Necessary Condition

$$\begin{cases} 2x+\lambda_2-\lambda_3 & =0 \\ 1+\lambda_1-\lambda_2-\lambda_3 & =0 \end{cases}$$

The first equation imposes $\lambda_2>0$ and thus $x=y$ It is therefore a matter of minimizing x^2+x for $-\frac{3}{2}\le x\le 0$ minimum at $x=y=-\frac{1}{2}$ and equals $-\frac{1}{4}$

4.

(1) The function is coercive and the constraint is closed.

(2)

$$g_1 = 2x-y^2-1, \quad g_2=-x$$

$$\nabla g_1 = \begin{pmatrix} 2 \\ -2y \end{pmatrix}, \quad \nabla g_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

If $I=\{1\}$ or $\{2\}$ We have $\nabla g_1 \neq 0$ and $\nabla g_2 \neq 0$ hence we have qualification in these two cases.

The case $I=\{1,2\}$ is impossible, we thus have qualification at every point.

(3)

$$\begin{cases} 0 = 2(x-2)-\lambda_1+2\lambda_2 \\ 0 = 2y-2y\lambda_2 \end{cases}$$

(4) We obtain $\lambda_2=1$ or $y=0$

Case $y=0$ Then $2x\ge 1$ hence $x-2<0$ and thus $\lambda_2>0$ We thus obtain $x=1/2$

Case $\lambda_2=1$ Then $2x-y^2=1$ We thus have $x>0$ and thus $\lambda_1=0$ We then obtain $x=1$ and thus $y=\pm 1$

(5) The minimum is thus 2 at $(1,\pm 1)$