Solution 1

1. Qualification using a non-linear constraint

$$\begin{cases} x^3 + y^3 - 3xy + 1 &= 0 \\ 3x^2 - 3y &= 0 \\ 3y^2 - 3x &= 0 \end{cases}$$

We obtain $x^4 = y^2 = x$ and thus the constraint is only unqualified at the point $x = y = 1$. The solution $(0, 0)$ does not satisfy the constraint.

Qualification in the general case We must show, as for the previous exercise, that ∇h is not null.

$$\nabla h = \begin{pmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{pmatrix}$$

It is not null unless $x^4 = x$. We return to the study of the previous case.

Necessary condition

$$\begin{cases} 1 + 3\mu x^2 - 3\mu y &= 0 \\ 1 + 3\mu y^2 - 3\mu x &= 0 \\ x^3 + y^3 - 3xy + 1 &= 0 \end{cases}$$

Subtracting the first and second equations, we get

$$y^2 - x^2 + y - x = 0,$$
$$y(1 + y) = x(1 + x)$$

Or the problem $a = x(1 + x)$ has only two solutions in terms of a

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}$$

which gives us two values for x and y of the form

$$-\frac{1}{2} \pm M$$

We substitute this result into the constraint. We separate according to the case where the two signs are the same or different.

Case $x = 0.5 + M$ **and** $y = 0.5 - M$. We then have

$$\left(-\frac{1}{2}+M\right)^3+\left(-\frac{1}{2}-M\right)^3-3\left(\frac{1}{4}-M^2\right)+1=0$$

This gives us

$$-\frac{1}{8}+\frac{3M}{4}-\frac{3M^2}{2}+M^3-\frac{1}{8}-\frac{3M}{4}-\frac{3M^2}{2}-M^3-\frac{3}{4}+3M^2+1=0$$

This simplifies to $0=0$ which is always true whatever M.

Case $x=y=0.5+M$. We obtain the equation on M

$$2M\left(M-\frac{3}{2}\right)^2=0$$

We then find that the point is unqualified except for the case $M=0$.

Conclusion The solution to the maximization problem is therefore 2 for $x=y=1$. We note that it is an unqualified point but it still verifies all the necessary conditions of Kuhn and Tucker.

2. (1) **Method using the non-linear constraint (avoid if possible)**

We consider the set of unqualified points. Let $(\lambda,\mu)\in\mathbb{R}_+^3\times\mathbb{R}$ such that

$$\begin{cases} 0= & -\lambda_1x_1-\lambda_2x_2-\lambda_3x_3 \\ 0= & \lambda_1\begin{pmatrix}-1\\0\\0\end{pmatrix}+\lambda_2\begin{pmatrix}0\\-1\\0\end{pmatrix}+\lambda_3\begin{pmatrix}0\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}x_2x_3\\x_1x_3\\x_1x_2\end{pmatrix}\end{cases}$$

The second equation allows us to obtain by equating term by term $\lambda_i=\mu x_jx_k$ with $i\neq j\neq k$.

We inject these equalities into the first equation to obtain

$$3\mu x_1x_2x_3=0$$

hence, since $x_1x_2x_3=2$, we have $\mu=0$. We thus obtain $\lambda=0$. Every point is thus qualified.

General Method

We have $I(x)=\emptyset$ since $x_i\neq 0$. We only need to show that $\{\nabla h\}$ is free, i.e., non-null.

$$\nabla h=\begin{pmatrix}x_2x_3\\x_1x_3\\x_1x_2\end{pmatrix}\neq 0 \text{ since } x_i\neq 0$$

Regarding maintaining the necessary conditions of the Kuhn and Tucker theorem for a maximum point.

$$\begin{cases} x_2 + 2x_3 - \lambda_1 + \mu x_2 x_3 & = 0 \\ x_1 + 2x_3 - \lambda_2 + \mu x_1 x_3 & = 0 \\ 2x_1 + 2x_2 - \lambda_3 + \mu x_1 x_2 & = 0 \end{cases}$$

with $\mu \in \mathbb{R}$, $\lambda_i \geq 0$ and the constraint $x_1x_2x_3 = 2$.

$x_1x_2x_3 = 2$ hence $x_i \neq 0$. We obtain by exclusion condition $\lambda_i = 0$. We find ourselves in the interior of the constraint.

We subtract the first and second equations, we get

$$(x_2 - x_1)(1 + \mu x_3) = 0$$

We have $\mu \neq -1/x_3$ because if we inject into the first equation, we would have $2x_3 = 0$ impossible. We thus have $x_1 = x_2$.

$$\begin{cases} x_1 - x_2 & = 0 \\ x_1 + 2x_3 + \mu x_1 x_3 & = 0 \\ 4x_1 + \mu x_1^2 & = 0 \\ x_1^2 x_3 - 2 & = 0 \end{cases}$$

From the third equation, we get $\mu = -4/x_1$ which, when substituted into the second equation, gives $x_1 = 2x_3$. We thus obtain

$$\begin{cases} x_1 & = \sqrt[3]{4} \\ x_2 & = \sqrt[3]{4} \\ x_3 & = \sqrt[3]{4}/2 \end{cases}$$

(2) It is said that under the constraint $x_1x_2x_3 = 2$, the function to be maximized is coercive. To do this, we change the variable x_3 to

$$x_3 = \frac{2}{x_1 x_2}$$

We thus obtain a new function

$$f(x) = x_1 x_2 + \frac{4}{x_1} + \frac{4}{x_2}$$

which is coercive. We can show this by minimizing with respect to x_1 while keeping the function fixed.

The minimum is reached when

$$x_2^{\min} = 2\sqrt{\frac{1}{x_1}}$$

for a value of

$$f(x_1, x_2^{\min}) = 2x_1\sqrt{\frac{1}{x_1}} + \frac{4}{x_1} + 2\sqrt{x_1}$$

which converges to infinity when x_1 converges to infinity.

The constraint being closed, the problem admits a solution.