Assignment 2

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric.
(1) Show that

$$m = \min_{\|x\|=1} \frac{1}{2} x^T A x$$

is the smallest eigenvalue of A.
(2) Let $\{v_i\}_{i=1,\ldots,k}$ be a family of orthogonal eigenvectors of A. Show that the quantity

$$\min_{\substack{\|x\|=1 \\ v_i^T x=0\ \forall i=1,\ldots,k}} \frac{1}{2} x^T A x$$

is an eigenvalue of A.

2. Let P be the hyperplane in $\mathbb{R}^N$ defined by the equation $c^T x = d$ (where $c \in \mathbb{R}^n$, $d \in \mathbb{R}$). Compute the orthogonal projection of a point y in $\mathbb{R}^n$ onto P, that is, the minimum of the problem

$$\min_{c^T x = d} \frac{1}{2} \|x - y\|^2 .$$

3. Do the following problems have a unique solution? If so, calculate the solution(s).
(1)

$$\begin{array}{ll} \min & x^2 + y^2 + 2z^2 \\ \text{subject to} & x + y \geq 1 \\ & x + 2y + z \geq 0 \\ & y \leq z \end{array} ;$$

(2)

$$\begin{array}{ll} \min & x^2 + y \\ \text{subject to} & y \leq 0 \\ & y \geq x \\ & x + y + 3 \geq 0 \end{array} .$$

4. Consider the problem

$$(P) \min_{(x,y)\in K} (x-2)^2 + y^2 \quad \text{where} \quad K = \{(x,y) \in \mathbb{R}^2 \mid 2x - y^2 \leq 1 \text{ and } x \geq 0\}.$$

(1) Show that the problem has at least one solution.
(2) Show that the constraint is qualified at every point.
(3) Write the necessary optimality conditions for the problem.
(4) Find all points satisfying the necessary optimality conditions.
(5) Deduce the solution(s) to problem (P).