THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4210 Financial Mathematics 2024-2025 Term 1

Homework Assignment 2

Due Date: 11:59PM, 17 October, 2024

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website `https://www.cuhk.edu.hk/policy/academichonesty/`

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

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Signature Date

General Regulations

- All assignments will be submitted and graded on CUHK Blackboard. You can view your grades and submit regrade requests here as well.
- Late assignments will receive a grade of 0.
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction.
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

Problem 1. Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion, $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = (x - K)_+$ for some constant $K > 0$, and $r > 0$, $T > 0$ be some positive constants.

a) Compute the value

$$u(t,x) := \mathbb{E}\big[e^{-r(T-t)} f(B_T) \big| B_t = x\big].$$

b) Use your results in Question a) to compute the partial derivatives $\partial_t u(t,x)$, $\partial_x u(t,x)$ and $\partial^2_{xx} u(t,x)$.

c) Check that the function $u : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is solution to the PDE

$$\partial_t u(t,x) + \frac{1}{2}\partial^2_{xx} u(t,x) - r u(t,x) = 0.$$

Problem 2. Let us consider a continuous time market, where the interest rate is $r > 0$, the risky asset $S = (S_t)_{0 \leq t \leq T}$ follows the Black- Scholes model with drift μ and volatility $\sigma > 0$.

a) Let $\mathbb{Q}$ denote the risk neutral probability, and $B^{\mathbb{Q}}$ denote the corresponding Brownian motion under $\mathbb{Q}$. Give the expression of S_t as a function of $(t, B^{\mathbb{Q}}_t)$.

b) We consider an option with payoff $\mathbf{1}_{\{S_T \geq K\}}$ for some strike $K > 0$. Compute

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-rT} \mathbf{1}_{\{S_T \geq K\}}\big].$$

c) Similarly, let us consider an option maturing at time T with payoff function

$$g(x) := \begin{cases} x & \text{if } x \leq K_1 \ , \\ \frac{K_1}{K_1 - K_2}(x - K_2) & \text{if } K_1 < x \leq K_2, \\ x - K_2 & \text{if } x > K_2 \ , \end{cases}$$

for strikes K_1, $K_2 > 0$. Compute the following expectation value

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-rT} g(S_T)\big].$$

Note: Please express the above expectation values in terms of the parameters S_0, r, μ, σ, T and K. You may use $\Phi : \mathbb{R} \longrightarrow [0,1]$ to denote the cumulative distribution function of the standard normal distribution $N(0,1)$.

Problem 3. Compute the price of a European call option written on a non-dividend-paying stock. The current stock price is \$100 and the volatility of the stock price is 30%. The maturity of the option is in nine months and the strike price is \$95. The risk-free interest rate with continuous compounding is 2% per annum. (Given: $\sigma = 0.3$, $K = 95$, $S_0 = 100$, $r = 0.02$ and the maturity period is 9 months, which is equivalent to $T = \frac{3}{4}$)

Note: You can use a calculator or computer software to compute the price and obtain the numerical values.