MATH4210: Financial Mathematics Tutorial 3

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Binomial Tree Models

Question

Given the current price of the underlying stock, $S(0) = 10$*. Each year, the stock price goes up or down by* $u = 1.2$ *and* $d = 0.8$*, respectively. The annual risk-free interest rate is* 5%*.*
a) Price a European put option maturing in two years from now, with exercise price $K = 10$*. Consider the discrete compound case.*
b) Consider (a), suppose the put option is American. What is its price?

Binomial Tree Models

Answer

Denote by $S_t, t = 0, 1, 2$ *the stock price at time now, 1 year later and two year later respectively. Similarly, we denote by* $P_t, t = 0, 1, 2$ *the put option price at different times. Clearly, at* $t = 1$, S_t *can take the values*

$$S_1 = uS_0 = 12 \text{ or } S_1 = dS_0 = 8.$$

Start from possible outcome of S_1 *(which means we assume we already observe* S_1*), the possible outcome of* S_2 *is given by:*

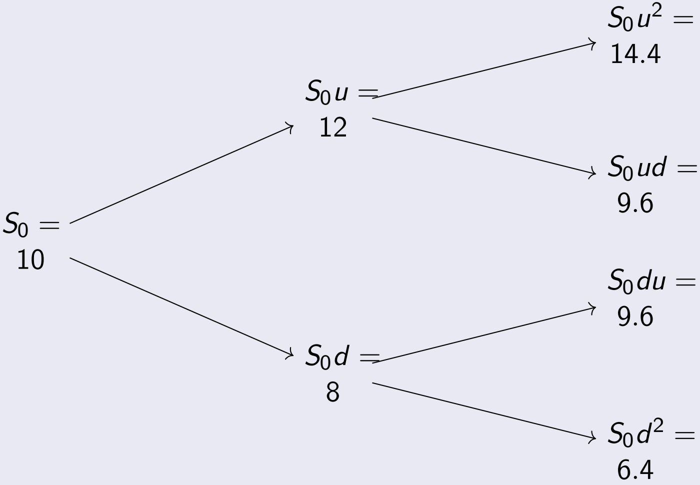
$$S_2 = uS_1 \text{ or } S_2 = dS_1$$

Then it's clear that S_2 *can take three values:*

$$S_2 = u^2 S_0 \text{ or } S_2 = udS_0 \text{ or } S_2 = d^2 S_0$$

Binomial Tree Models

Answer



Answer

The next step is to compute the EMM (probability q). Since we have two periods and each period takes one year, $\Delta t = 1$. According to the formula:

$$q = \frac{1 + r\Delta t - d}{u - d} = 0.625$$

Finally we do backward computation of option price. Recall the payoff function of European put option is $g(x) := (K - x)_+$. Then the option price at maturity is:

$$P_2 = g(S_2) = (K - S_2)_+$$

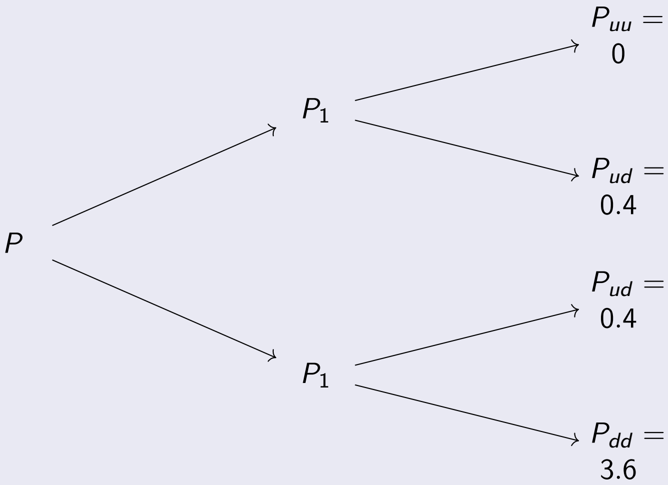
Since S_2 can take 3 values, then option price can accordingly take three values:

$$P_2 = (K - u^2 S_0)_+ = 0 \text{ or } P_2 = (K - udS_0)_+ = 0.4 \text{ or } P_2 = (K - udS_0)_+ = 3.6$$

We denote the three cases by P_{uu}, P_{ud} and P_{dd} respectively.

Binomial Tree Models

Answer



Answer

Then, we apply the formula to compute P_1 *using the last two smaller trees, if* $S_1 = uS_0$*:*

$$P_1 = (1 + r\Delta t)^{-1}[qP_{uu} + (1-q)P_{ud}] \approx 0.14$$

and if $S_1 = dS_0$*:*

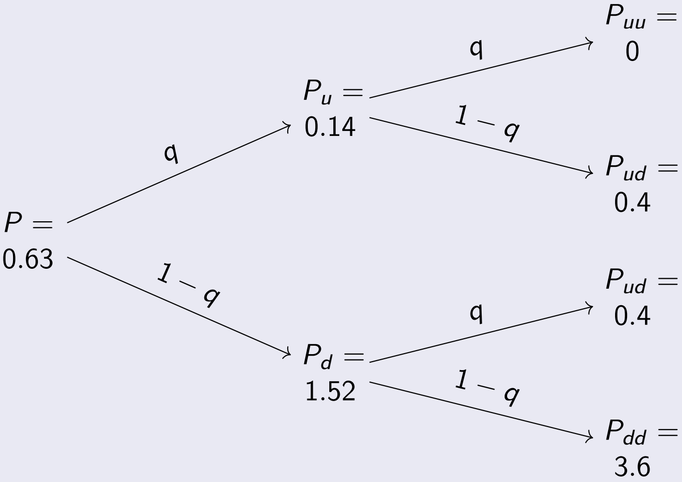
$$P_1 = (1 + r\Delta t)^{-1}[qP_{ud} + (1-q)P_{dd}] \approx 1.52$$

denoted by P_u *and* P_d *respectively. Finally, we apply the formula again:*

$$P = (1 + r\Delta t)^{-1}[qP_u + (1-q)P_d] \approx 0.63$$

Binomial Tree Models

Answer



Binomial Tree Models

The main difference between American option and European option is that owners of American options are free to exercise the option at anytime before expiry. If either the seller or buyer stupid enough, the buyer can exercise immediately after buying this option.
The key intuition is the owner needs to do a little bit math to decide whether to exercise immediately or wait. Therefore, the payoff function logically becomes:

$$g(x,t) = max\{(K-x)_+, P_t\}$$

where the former means at time t, the money he will receive if he exercises immediately. And the latter means at the same time, the option value if he holds this option till next time node.
Note in the binomial tree model, we only consider finitely many time nodes. And the buyer are assumed to be able to exercise on those nodes.

Answer (b)

Denote by P_t^A *the American put option price. Note that at time* $t = 2$, $P_2^A = P_2 = (K - S_2)_+$, *which means it's the same if either exercise immediately or wait (it's about to expire, you don't have any other choice). Moreover,* q *does not change. What we need to do is to compute the possible values of exercising immediately at different time and cases. Denote*

$$(K - uS_0)_+ = 0 \text{ and } (K - dS_0)_+ = 2$$

Then

$$P_1^A = max\{(K - uS_0)_+, P_u\} = P_u = 0.14 \text{ if } S_1 = uS_0$$

and

$$P_1^A = max\{(K - dS_0)_+, P_d\} = (K - dS_0)_+ = 2 \text{ if } S_1 = dS_0$$

Answer

By applying the formula,

$$P = (1 + r\Delta t)^{-1}[qP_u^A + (1-q)P_d^A] \approx 0.80$$

and

$$P = max\{(K - S_0)_+, P\} = P = 0.80$$

It's clear at $t = 2$ the value of American option equals that of European option. But at $t = 1$, if S_1 goes up, we choose to hold the option till next time. However, if S_1 goes down, we exercise immediately. Finally, at $t = 0$ we hold till next year and see what happens.

Question

Consider a sequence of i.i.d. random variables $\{\xi_k\}_{k\in\mathbb{N}^*}$ *which takes value* u *with probability* q *and* d *with probability* $1-q$ *with* q *the risk neutral probability. Then the stock price can be written as* $S_n = S_0\Pi_{k=1}^n \xi_k$. *Show that the discounted stock price is a discrete martingale.*

Answer

First, we need to prove the integrability. Fix $n > 0$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[|(1+r)^{-n}S_n|] &= (1+r)^{-n}\mathbb{E}^{\mathbb{Q}}[S_0\Pi_{k=1}^n\xi_k] \\ &= (1+r)^{-n}S_0\Pi_{k=1}^n\mathbb{E}^{\mathbb{Q}}[\xi_k] \\ &= (1+r)^{-n}S_0(qu+d-qd)^n \\ &< \infty\end{aligned}$$

Answer

Second, for $n > 0$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(1+r)^{-(n+1)}S_{n+1}|\mathcal{F}_n] &= \mathbb{E}^{\mathbb{Q}}[(1+r)^{-n}S_n * (1+r)^{-1}\xi_{n+1}|\mathcal{F}_n] \\
&= (1+r)^{-n}S_n * (1+r)^{-1}\mathbb{E}^{\mathbb{Q}}[\xi_{n+1}|\mathcal{F}_n] \\
&= (1+r)^{-n}S_n * (1+r)^{-1}\mathbb{E}^{\mathbb{Q}}[\xi_{n+1}] \\
&= (1+r)^{-n}S_n * \frac{qu+(1-q)d}{1+r} \\
&= (1+r)^{-n}S_n * \frac{(1+r)(u-d)}{(1+r)(u-d)} \\
&= (1+r)^{-n}S_n
\end{aligned}$$

Hence, the discounted stock price is a discrete martingale under probability measure $\mathbb{Q}$.

Remark

According to the martingale property of discounted option price, we have:

$$\begin{aligned} f &= \mathbb{E}^{\mathbb{Q}}[(1+r\Delta t)^{-2} f_2 | \mathcal{F}_0] \\ &= (1+r\Delta t)^{-2}\{\mathbb{Q}[f_{uu}]f_{uu} + \mathbb{Q}[f_{ud}]f_{ud} + \mathbb{Q}[f_{dd}]f_{dd}\} \\ &= (1+r\Delta t)^{-2}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}) \end{aligned}$$

Question

Consider the question (a) at the beginning, compute the call option price with same settings. What's the relation between European call and European put?

Answer

According to the remark, the call price is:

$$f = (1 + r\Delta t)^{-2}(q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}) \approx 1.56$$

Compute $f - P$ *and* $S_0 - (1 + r\Delta t)^{-2}K$

$$f - P = 1.56 - 0.63 = 0.93$$

and

$$S_0 - (1 + r\Delta t)^{-2}K = 10 - 1.05^{-2} * 10 \approx 0.93$$

Why?
At $t = 2$, $f_2 - P_2 = S_2 - K$. *And discounted option prices, stock price are martingales. Therefore, we have the following equality by direct computation:*

$$f - P = S_0 - (1 + r\Delta t)^{-2}K$$

Brownian Motion

Question

Calculate $E(W_sW_t)$ *for* $s, t \in [0, \infty)$ *for a Wiener process* $(W_t)_{t\geq 0}$

Answer

Let s denote the smaller between s and t. Then $E(W_sW_t) = E\{W_s(W_t - W_s + W_s)\} = E\{W_s(W_t - W_s)\} + EW_s^2$. *By the independence of* $W_s = W_s - W_0$ *and* $W_t - W_s$*, we have* $E\{W_s(W_t - W_s)\} = EW_sE(W_t - W_s) = 0$. *Hence,* $E(W_sW_t) = EW_s^2 = s = \min(s, t)$.