MATH4210: Financial Mathematics Tutorial 1

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Definition (Normal Distribution)

Given a real-valued random variable $X : \Omega \to \mathbb{R}$, it follows the normal distribution with parameters μ, σ if the probability density function (pdf) of X is given by $f : \mathbb{R} \to \mathbb{R}_+$:

$$\forall x \in \mathbb{R}, f(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

It always denoted as $X \sim N(\mu, \sigma^2)$.

Exercise

Given $X \sim N(\mu, \sigma^2)$, compute the following:
(a). $\mathbb{E}(X), \mathbb{E}(X^2), Var(X)$;
(b). $\mathbb{E}(e^{tX})$
(c). $\mathbb{E}(e^{itX})$ for t fixed (Characteristic Function).

Note that $f(t) := \mathbb{E}(e^{tX})$ is called the Moment Generating Function.

Solution for (a)

For a random variable $X \sim N(\mu, \sigma^2)$, the expectation $E(X)$ is defined as: $\mathrm{E(X)} = \int_{-\infty}^{\infty} x f_X(x)\, dx$, where the probability density function (PDF) of the normal distribution $N(\mu, \sigma^2)$ is: $\mathrm{f}_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

Thus, $\mathrm{E(X)} = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$.

To simplify the integral, we perform a variable substitution $z = \frac{x-\mu}{\sigma}$, so that $x = \sigma z + \mu$. The integral then becomes:

$\mathrm{E(X)} = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz + \sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$.

The first integral is the standard normal distribution integral, which equals 1: $\mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = \mu$.

The second integral is the integral of an odd function $z \cdot e^{-z^2/2}$ over a symmetric interval, which equals 0: $\sigma \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = 0$.

Thus, the expectation is: $\mathrm{E(X)} = \mu$.

Solution for (a)

The second moment $E(X^2)$ is defined as:
$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$
To simplify the integral, we perform a variable substitution $z = \frac{x-\mu}{\sigma}$, so that $x = \sigma z + \mu$. The integral then becomes:
$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} (\sigma z + \mu)^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$
Expanding $(\sigma z + \mu)^2$, we get:
$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} \left(\sigma^2 z^2 + 2\sigma z\mu + \mu^2\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$
We can break this into three integrals:
$\mathrm{E}(\mathrm{X}^2) = \sigma^2 \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz + 2\sigma\mu \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$

1. The second integral is zero because it's an odd function.
2. The third integral equals μ^2:
3. The first integral can use integration by parts:

$\int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = 1$. Thus, we have: $\mathrm{E}(\mathrm{X}^2) = \sigma^2 + \mu^2.$

Solution for (a)

Therefore, $Var(X) = E(X^2) - (E(X))^2 = \sigma^2$

Solution for (b)

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$$

$$z = \frac{x-\mu}{\sigma}, \text{so that } x = \sigma z + \mu.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(t(\sigma z + \mu) - \frac{z^2}{2}\right) dz.$$

$$= \exp(\mu t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(t\sigma z - \frac{z^2}{2}\right) dz.$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{(z-\sigma t)^2}{2}) dz.$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

Solution for (c)

$$
\begin{aligned}
E(e^{itX}) &= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx. \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(itx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx. \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}\left[(x-\mu-i\sigma^2 t)^2 + \sigma^4 t^2 - 2\mu\sigma^2 it\right]\right) dx. \\
&= \exp\left(\mu it - \frac{1}{2}\sigma^2 t^2\right) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}\left[(x-\mu-i\sigma^2 t)^2\right]\right) d \\
&= \exp\left(\mu it - \frac{1}{2}\sigma^2 t^2\right)
\end{aligned}
$$

Discrete Distribution

Exercise

Determine the expectation and the variance of the following discrete random variables:

(a) **Discrete uniformly distributed random variable**:
$\mathrm{P}_X(k) = \frac{1}{n}, \quad k = 1, 2, \ldots, n.$

(b) **Bernoulli random variable with parameter** p:
$\mathrm{P}_X(1) = p, \quad p_X(0) = 1 - p, \quad 0 < p < 1.$

(c) **Binomial random variable with parameters** (n, p):
$\mathrm{P}_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n,$
where $n \geq 1$ is an integer and $0 < p < 1$.
(The number of successes in n independent trials.)

(d) **Geometric random variable with parameter** p:
$\mathrm{P}_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots,$
where $0 < p < 1$.
(The number of trials until the first success.)

Solution for (a) and (b)

For $Var(X)$, the variance of X we have: Var(X) := E(X - EX)$^2 = EX^2 - (EX)^2$.

(a)
- $EX = \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$.
- $EX^2 = \sum_{i=1}^{n} \frac{i^2}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$.
- $Var(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)(4n+2-(3n+3))}{12} = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$.

(b)
- $EX = 1 \times p + 0 \times (1-p) = p$
- $EX^2 = p$
- $Var(X) = p - p^2 = p(1-p)$

Solution for (c)

(c)

- $$\begin{aligned} EX &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\ &= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l (1-p)^{(n-1)-l} \\ &= np. \end{aligned}$$

Solution for (c)

- $$\begin{aligned} EX^2 &= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=2}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k} + np \\ &= n(n-1)p^2 + np. \end{aligned}$$
- $$\begin{aligned} Var(X) &= n(n-1)p^2 + np - n^2p^2 \\ &= np(1-p). \end{aligned}$$

Hint Another solution: If $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p, then $X := X_1 + X_2 + \cdots + X_n$ is a Binomial random variable with parameter (n, p).

Solution for (d)

(d) •

$$
\begin{aligned}
(d) \quad EX &= \sum_{k=1}^{\infty} k(1-p)^{k-1} p \\
&= p \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} (1-p)^{k-1} \\
&= p \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{1-(1-p)} \\
&= \sum_{n=1}^{\infty} (1-p)^{n-1} = \frac{1}{p}.
\end{aligned}
$$

Solution for (d)

- $$EX^2 = \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p$$
  $$= \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1}p + \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$
  $$= p(1-p)\frac{d^2}{dp^2}f(p) + \frac{1}{p}$$
  $$= \frac{2(1-p)}{p^2} + \frac{1}{p},$$
  where $f(p) := \sum_{k=0}^{\infty}(1-p)^k$.

  $$Var(X) = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$
  $$= \frac{1-p}{p^2}.$$

Exercise

Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables. For any $j \in \mathbb{N}, \mathbb{P}(Y_j = \pm 1) = \frac{1}{2}$. Define for $n \in \mathbb{N}, X_n = \sum_{j=1}^{n} Y_j$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Solution:
In order to prove that (X_n) is a martingale, we are going to verify by definition.
1. Fix $n \in \mathbb{N}$.

$$\begin{aligned}\mathbb{E}(|X_n|) &= \mathbb{E}(|\sum_{j=1}^{n} Y_j|) \\ &\leq \sum_{j=1}^{n} \mathbb{E}(|Y_j|) \\ &= n(1 * \frac{1}{2} + |-1| * \frac{1}{2}) \\ &= n < \infty\end{aligned}$$

2. Fix $n \in \mathbb{N}$, denote $\mathcal{F}_n = \sigma(X_0, ..., X_n)$.

$$\begin{aligned} \mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_n + Y_{n+1}|\mathcal{F}_n) \\ &= \mathbb{E}(X_n|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}|\mathcal{F}_n) \\ &= X_n + \mathbb{E}(Y_{n+1}) \\ &= X_n \end{aligned}$$

By 1 and 2, (X_n) is a martingale.

Remark

It still works when $\mathbb{P}(Y_j = 2) = \frac{1}{3}$ and $\mathbb{P}(Y_j = -1) = \frac{2}{3}$. (X_n) will still be a martingale as long as the expectation is 0 (Exercise!).