MATH4210: Financial Mathematics Tutorial 11

Yi Shen

The Chinese University of Hong Kong

yishen@math.cuhk.edu.hk

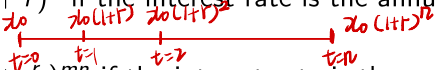
27 November, 2024

Interest Rate

Let r be the interest rate. Suppose that you place $\$x_0$ in an account in a bank. After n years, you will have the amount

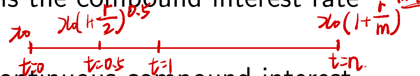
$\frac{1}{1+nr}$

- $y_n = x_0(1 + nr)$ if the interest rate is the simple interest rate.
- $y_n = x_0(1 + r)^n$ if the interest rate is the annual compound interest rate.



$\frac{1}{(1+r)^n}$

- $y_n = x_0(1 + \frac{r}{m})^{mn}$ if the interest rate is the compound interest rate and compound m times per annul.



- $y_n = x_0 e^{nr}$ if the interest rate is the continuous compound interest rate.

$\frac{1}{1+(\frac{r}{m})^{mn}}$

$\frac{1}{e^{nr}}$

Interest Rate

Q1 (a)

$1+$

$y_n = x_0(1+nr)$

$100 = x_0(1+10\times 0.02) \Rightarrow x_0 = \frac{100}{(1+10\times 0.02)}$

(b)

$y_n = x_0 e^{nr}$

$y_n = 100\, e^{10\times 0.03}$

Question

a) Find the value of a 10*-year zero-coupon bond of face value* \$100 *if the the annual simple interest rate is* 2%*.*

b) Find the face value of a 10*-year zero-coupon bond if it is issued for* \$100 *and the continuous compound interest rate is* 3%*.*

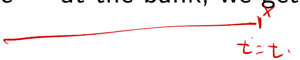
Present Value

Since we can always use $\$x_0$ now as principal in a risk-free investment at (continuous compound interest) rate $r > 0$ guaranteeing the amount

$$y_t = x_0 e^{rt} > x_0$$

x0

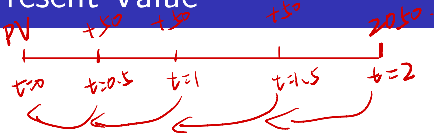
at time t. Equivalently, if we deposit $\$xe^{-rt}$ at the bank, we get $\$x$ at time t, thus



We call xe^{-rt} the present value (PV) of x,

which is also called the discounted value of x at the future time t, and the factor e^{-rt} is called the discount factor.

Present Value



$$PV = \sum_{n=1}^{4} 50 e^{-r(0.5n)} + 2000 e^{-2r}$$

$$= 50 e^{-0.5\times r} + 50 e^{-1\times r} + \cdots$$

$$+ 2050 e^{-2r}.$$

Question

Pricing a coupon bond: consider a 2*-year* \$2000 *bond, that has coupons every* 1/2 *year in the amount of* \$50*, for a total of four times until* 2 *years at which time you receive* \$2050*. Suppose the continuous compound interest rate is* r*. What is its price of the bond?*

Annuity/Perpetual Bond

$$PV = \sum_{i=1}^{\infty} 10,000 \times \frac{1}{(1+0.025)^i} = 10,000\left(\frac{1}{1-\frac{1}{1.025}}\right)$$

Question

An imaginary nice government that does not exists on this planet promises to pay you (and your descendant) \$10,000 *immediately and the same amount every year perpetually. If the the compound annual interest rate is* 2.5%, *what is its present value of this plan?*

Annuity/Perpetrial Bond

Plan 1 : $PV_1 = \sum_{i=1}^{\infty} \frac{1000}{(1+2\%)} = 1000 \times \frac{1}{1-\frac{1}{1.02}}$

plan 2 : $PV_2 = 50,000$

if $PV_1 \geqslant PV_2 \Rightarrow$ Joyce accept this offer

Question

Joyce wants to use a land to build a church. The government requires she to pay the nominal rent 1,000 *every year perpetually. A bank offer a plan: Joyce pay the bank* 50,000 *at once and the bank promises to pay* 1,000 *to the government every year. Suppose the discrete annual compound interest is* 2%. *Should Joyce accept this offer? (Unit: \$)*

Options Revisit

if we assume C_1 & C_3 are well-defined.

Then $C_1 + C_3 = 17 \geq 2C_2 = 18$. ×

$\Rightarrow C_2(K=100) \leq 8.5$ $\Rightarrow$ C_2 is higher than the actual value

Question (Example on Slides 5)

$\pi(t) = C_1 + C_3 - 2C_2$

(a).Suppose that we have three European call options with the same maturity T in the financial market whose price at time $t = 0$ are:

$$C_1(K = 90) = 10$$
$$C_2(K = 100) = 9$$
$$C_3(K = 110) = 7.$$

Suppose the interest rate is zero. Construct the arbitrage strategy.

(b). At $t = 0$, the underlying asset $S_0 = 100$. We keep C_1 and C_3 the same. But We don't have C_2, instead there is a European put option with the same setting such that $P_2(K = 100) = 9$. Find the arbitrage strategy.

long $C_E(90) = 10$ long $C_E(110) = 7$ short $2P_E(100) = 9$ short 2 stocks.

Options Revisit

(b) $C(K=100) \leq 8.5$

$C_2(K=100) - P_2(K=100) = S_0 - K \Rightarrow P_2(K=100) \leq 8.5$

Question (Example on Slides 5)

(a).Suppose that we have three European call options with the same maturity T in the financial market whose price at time $t = 0$ are:

$$C_1(K = 90) = 10$$
$$C_2(K = 100) = \boxed{9}$$
$$C_3(K = 110) = 7.$$

$-17+18+200$

Suppose the <u>interest rate is zero.</u> Construct the arbitrage strategy.
(b). At $t = 0$, the underlying asset $S_0 = 100$. We keep C_1 and C_3 the same. But We don't have C_2, instead there is a European put option with the same setting such that $P_2(K = 100) = 9$. Find the arbitrage strategy.

long C_1 & C_3 short $2P_2(K=100)$, short 2 Stocks   initial $= +201$

if $S_T \le 90$ net cash flow $= 201 - 2(100 - S_T) - 2S_T$

$= 201 - 200 + 2S_T - 2S_T$

$= 1$

if $90 < S_T \le 100$ net cash flow $= 201 - 2(100 - S_T) + (S_T - 90) - 2S_T$

$= \underline{201} - \underline{200} + \utilde{2S_T} + \utilde{S_T} - \underline{90} - 2S_T$

$= S_T - 89 > 0$

if $100 < S_T \le 110$ net cash flow $= 201 + (S_T - 90) - 2S_T$

$= \underline{201} + S_T \underline{-90} - 2S_T$

$= 111 - S_T > 0$

if $S_T > 110$ net cash flow $= 201 + (S_T - 90) + (S_T - 110) \cancel{-2}$

$-2S_T$

$= 201 - 90 - 110$

$= 1 > 0$