Q1

(1) $df(t, B_t) = d(tB_t) = B_t dt + t dB_t$

integrating both sides

$$tB_t - 0 = \int_0^t B_s ds + \int_0^t s dB_s$$

$$\Rightarrow tB_t = \int_0^t s dB_s + \int_0^t B_s ds$$

(2) from (1), we can get $tB_t = \int_0^t s dB_s + \int_0^t B_s ds$

$$\int_0^t B_s ds = tB_t - \int_0^t s dB_s$$

$$= \int_0^t t dB_s - \int_0^t s dB_s \quad (\text{since } B_0 = 0)$$

$$= \int_0^t (t-s) dB_s$$

(3) $E[\int_0^t B_s ds] = \int_0^t E[B_s] ds = 0$

$$var[\int_0^t B_s ds] = var[\int_0^t (t-s) dB_s]$$

$$= E[(\int_0^t (t-s) dB_s)^2] - (E[\int_0^t (t-s) dB_s])^2$$

$$= \int_0^t (t-s)^2 ds - 0$$

$$= \int_0^t t^2 + s^2 - 2ts \, ds$$

$$= t^2 s + \frac{1}{3}s^3 - ts^2 \Big|_0^t$$

$$= t^3 + \frac{1}{3}t^3 - t^3 = \frac{1}{3}t^3$$

Q2 $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$

(a) $d\pi_t^{x,\phi} = (\pi_t^{x,\phi} - \phi_t S_t) r \, dt + \phi_t (\mu S_t \, dt + \sigma S_t dB_t$

$= (r\pi_t^{x,\phi} + (\mu - r)\phi_t S_t) dt + \sigma \phi_t S_t \, dB_t$

$\Rightarrow \alpha_t = r\pi_t^{x,\phi} + (\mu - r)\phi_t S_t$

$\beta_t = \sigma \phi_t S_t$

(b) Recall the Girsanov theorem,

$$B_t^Q = B_t + \int_0^t \frac{\mu - r}{\sigma} ds = B_t + \frac{\mu - r}{\sigma} t$$

then we have

$$dB_t^Q = dB_t + \frac{\mu - r}{\sigma} dt$$

Then. we use this to substitute into original SDE:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$$

$$= \mu S_t \, dt + \sigma S_t \left(dB_t^Q - \frac{\mu - r}{\sigma} dt\right)$$

$$= r S_t \, dt + \sigma S_t \, dB_t^Q$$

therefore we can get

$$S_t = S_0 \exp[(r - \tfrac{1}{2}\sigma^2)t + \sigma B_t^Q]$$

$$= \exp[(r - \tfrac{1}{2}\sigma^2)t + \sigma B_t^Q]$$

(c) ① compute the value $V_0 = E^Q[e^{-rT} S_T^3]$

$$V_0 = E^Q[e^{-rT} S_T^3] = E^Q[e^{-rT} \exp(3(r - \tfrac{1}{2}\sigma^2)T + 3\sigma B_T^Q]$$

$$= e^{-rT + 3rT - \frac{3}{2}\sigma^2 T} E^Q[e^{3\sigma B_T^Q}]$$

$$= e^{2rT - \frac{3}{2}\sigma^2 T} (e^{\frac{9}{2}\sigma^2 T}) = e^{2rT + 3\sigma^2 T}$$

Since $B_T^Q \sim N(0, T)$

$3\sigma B_T^Q \sim N(0, 9\sigma^2 T)$

$E[e^X] = e^{\mu + \frac{1}{2}\sigma^2}$ if $X \sim N(\mu, \sigma^2)$

① $v(t,x) = x^3 \exp((2r+3\sigma^2)(T-t))$

firstly, $v(T,x) = x^3$

$$\partial_t v(t,x) = v(t,x)(-2r-3\sigma^2)$$

$$\partial_x v(t,x) = 3x^2 \exp((2r+3\sigma^2)(T-t))$$

$$\partial^2_{xx} v(t,x) = 6x \exp((2r+3\sigma^2)(T-t))$$

$$\partial_t v(t,x) + rx\partial_x v(t,x) + \tfrac{1}{2}\sigma^2 x^2 \partial_{xx} v(t,x) - rv(t,x)$$

$$= v(t,x)(-2r-3\sigma^2) + 3r\,v(t,x) + 3\sigma^2 v(t,x) - r v(t,x)$$

$$= v(t,x)(-2r-3\sigma^2+3r+3\sigma^2-r)$$

$$= 0$$

② $\tilde{S}_t = e^{-rt} S_t$

$$d\tilde{S}_t = -r\tilde{S}_t\,dt + e^{-rt}\,dS_t = -re^{-rt}S_t\,dt + re^{-rt}S_t\,dt + \sigma e^{-rt} S_t\,dB_t^Q = \sigma e^{-rt} S_t\,dB_t^Q$$

Apply the integral from of the ~~integ~~ Ito formula to $e^{-rt} v(t,S_t)$

$$e^{-rT}S_T^3 = e^{-rT}v(T,S_T) = e^{-r\cdot 0}v(0,S_0) + \int_0^T \partial_t(e^{-rt}v)\,dt + \int_0^T \partial_x(e^{-rt}v)\,dS_t + \tfrac{1}{2}\int_0^T \partial^2_{xx}(e^{-rt}v)(dS_t\,dS_t)$$

$$= v(0,S_0) + \int_0^T -re^{-rt}v + e^{-rt}\partial_t v\,dt + \int_0^T e^{-rt}\partial_x v(r\cancel{S_t} x)\,dt + \int_0^T e^{-rt}\partial_x v\,\sigma S_t\,dB_t^Q + \tfrac{1}{2}\int_0^T e^{-rT}\partial^2_{xx}v(\sigma^2 \cancel{S_t^2}x^2)\,dt$$

$$= v(0,S_0) + \int_0^T e^{-rt}(\partial_t v + rx\,\partial_x v + \tfrac{1}{2}\sigma^2 x^2 \partial^2_{xx}v - rv)\,dt + \int_0^T e^{-rt}\,\sigma x\,\partial_x v\,dB_t^Q$$

$$= v(0,S_0) + \int_0^T e^{-rT}\,\sigma x\,\partial_x v\,dB_t^Q$$

$$= v(0,S_0) + \int_0^T e^{-rT} S_t\,\sigma(\partial_x v)\,dB_t^Q$$

$$= v(0,S_0) + \int_0^T \partial_x v\,d\tilde{S}_t$$

$$= \exp((2r+3\sigma^2)T) + \int_0^T \partial_x v\,d\tilde{S}_t$$

$$= V_0 + \int_0^T \partial_x v\,d\tilde{S}_t$$

Therefore, according to the delta-hedging.

we can construct a self-financing portfolio:

π_t ~~denots~~ denotes the total wealth

ϕ_t : number of stocks

$\pi_t - \phi_t S_t$: wealth invested in the risk-free asset.

π_t is self-financing since $d\tilde{\pi}_t = de^{-rt}\pi_t = \pi_0 + \int_0^t \phi_s \, d(e^{-rt} S_t)$

$$= \pi_0 + \int_0^t \phi_s \, d\tilde{S}_t$$

then we let $\pi_T = g(S_T) = S_T^3$

$$\tilde{\pi}_T = e^{-rT} S_T^3$$

Finally, we can get dynamic trading strategy $\phi_t = \partial_x V(t, S_t)$

$$= 3S_t^2 \exp[(2r+3\sigma^2)(T-t)]$$

Therefore, the no-arbitrage option price at initial time 0 is $\pi_0 = V_0 = \exp[2rT + 3\sigma^2 T]$

~~It follows that~~ $\cancel{\tilde{\pi}_t = e^{-rt} V(t, S_t) \Leftrightarrow \pi_t = V(t, S_t)}$

Q3 :

construct 2 portfolio :

1 portfolio : at time t, long a call option
short a put option

so that $\pi_1(t) = C_E(t, K) - P_E(t, K)$

and $\pi_1(T) = (S_T - K)_+ - (K - S_T)_+ = S_T - K$

2 portfolio : at time $t \le T_D$, long a stock
borrow $Ke^{-rT} + De^{-rT_D}$

$\pi_2(t) = \cancel{S(t)}\ S_t - Ke^{-r(T-t)} - De^{-r(T_D - t)}$

$\pi_2(T_D) = S_{T_D} - Ke^{-r(T-T_D)} - D + D = S_{T_D} - Ke^{-r(T-T_D)}$

$\pi_2(T) = S_T - K$

$\pi_1(T) = \pi_2(T) = S_T - K$

$\Rightarrow \pi_1(0) = \pi_2(0)$

$\Rightarrow C_E(0, K) - P_E(0, K) = S_0 - Ke^{-rT} - De^{-rT_D}$

Q4 $t=0$

short ~~a~~ 2 Call $C_E(100,1)=11$

long a put $P_E(110,2)=14$

long a Put $P_E(90,2)=6$

long 2 stocks at $S_0=100$

net cash flow $=22-6-14-200=-198$

at $t=1$

if $S_1 \leq 100$ call option would not exercise

net cash flow $=-198$

if $S_1 > 100$ call options exercise and we sell 2 stocks

net cash flow $=-198-2(S_1-100)+2S_1$

$=-198+200=2$

at $t=2$, $S_1>100$

if $S_2 \leq 90$, $P_E(110,2)$ and $P_E(90,2)$ would exercise

net cash flow $=2+(110-S_2)+(90-S_2)$

$=202-2S_2>0$

if $90<S_2\leq 110$ $P_E(110,2)$ would exercise

net cash flow $=2+(110-S_2)=112-S_2>0$

if $S_2>110$, ~~all~~ both put option would not exercise

net cash flow $=2>0$

at $t=2$ $S_1 < 100$

if $S_2 \le 90$, put $P_E(90,2)$ and $P_E(110,2)$ would exercise, and sell 2 stocks

$$\text{net cash flow} = -198 + (90 - S_2) + (110 - S_2) + 2S_2$$
$$= -198 + 200 = 2 > 0$$

if $90 < S_2 \le 110$ $P_E(110,2)$ would exercise and sell 2 stocks

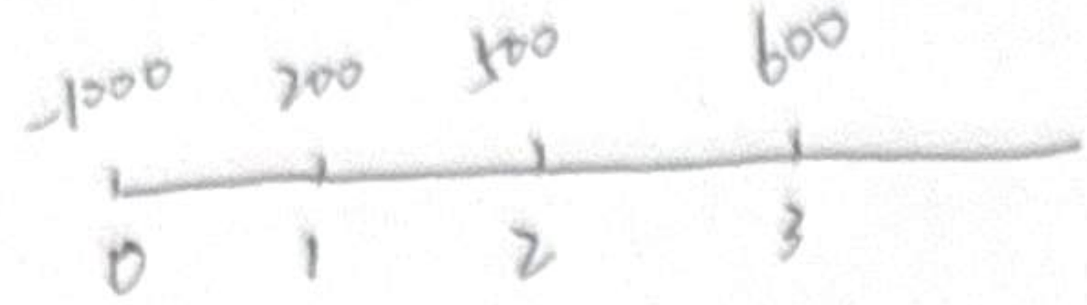
$$\text{net cash flow} = -198 + (110 - S_2) + 2S_2$$
$$= S_2 - 88 > 0$$

if $S_2 > 110$ both ~~exerci~~ put would not exercise, sell 2 stocks

$$\text{net cash flow} = -198 + 2S_2 > 0$$

Q5

Investment 1.



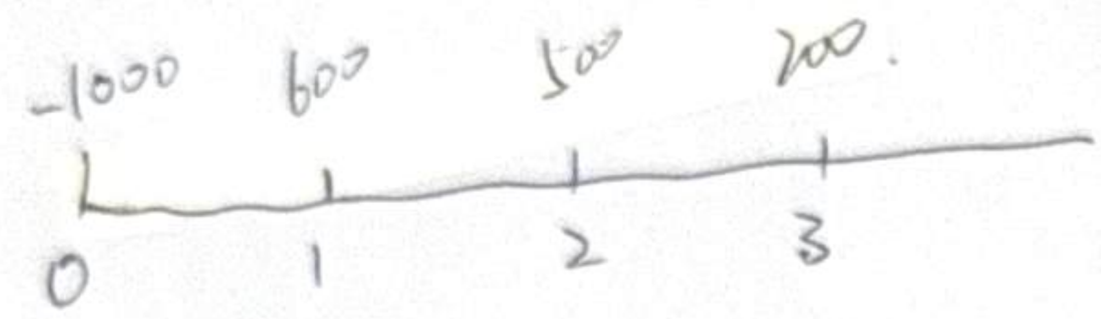
$$NPV_1 = \sum_{i=0}^{3} x_i (1+r)^{-i}$$

$$= -1000 + 200 \times (1+0.01)^{-1} + 500 \times (1+0.01)^{-2} + 600 \times (1+0.01)^{-3}$$

$$\approx -1000 + 198.02 + 490.148 + 582.354$$

$$= 270.522$$

Investment 2:



$$NPV_2 = -1000 + 600 \times 1.01^{-1} + 500 \times 1.01^{-2} + 200 \times 1.01^{-3}$$

$$= -1000 + 594.059 + 490.148 + 194.118$$

$$= 278.325$$

Investment 2 has a slightly higher NPV than investment 1, making it the better choice based on NPV criterion.