Math 4210 Assignment 3

Due time: November 28th, 2024, 23:59

Question 1

1. Let $f(t,x) := tx$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, apply Itô's formula on $f(t, B_t)$ to prove that
$$tB_t = \int_0^t s\ dB_s + \int_0^t B_s\ ds.$$
2. Deduce from above result that
$$\int_0^t B_s\ ds = \int_0^t (t-s)\ dB_s.$$
3. Compute
$$\mathbb{E}\Big[\int_0^t B_s\ ds\Big] \quad \text{and} \quad \mathrm{Var}\Big[\int_0^t B_s\ ds\Big].$$

Question 2

We consider a continuous time market, where the interest rate $r \geq 0$, and the risky asset $S = (S_t)_{0 \leq t \leq T}$ follows the Black-Scholes model with initial value $S_0 = 1$, drift μ and volatility $\sigma > 0$ (without any dividend), so that
$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma B_t\big),\ t \geq 0.$$

a) A self-financing portfolio is given by (x, ϕ), where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $(\Pi_t^{x,\phi})_{t\in[0,T]}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in form of
$$d\Pi_t^{x,\phi} \ = \ \alpha_t dt + \beta dB_t, \quad \text{for some (to be found) process } (\alpha, \beta).$$

b) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$.

Please give the expression of the process S_t as a function of $(t, B_t^{\mathbb{Q}})$.

c) We first consider an option with payoff $g(S_T) = S_T^3$ at maturity T.

- Compute the value
$$V_0 \ := \ \mathbb{E}^{\mathbb{Q}}\big[e^{-rT} S_T^3\big].$$

- Let $v(t,x) := x^3 \exp\big((2r+3\sigma^2)(T-t)\big)$, compute $\partial_t v$, $\partial_x v$ and $\partial^2_{xx} v$, and then check that v satisfies the equation
$$\partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) - rv(t,x) \ = \ 0, \qquad v(T,x) = x^3.$$

- Let $\tilde{S}_t := e^{-rt} S_t$ for all $t \geq 0$. Notice that both S_t and $\tilde{S}_t$ are functions of (t, B_t), apply the Itô formula on $e^{-rt} v(t, S_t)$ to deduce a process $(\phi_t)_{t \in [0,T]}$ such that
$$e^{-rT} S_T^3 = V_0 + \int_0^T \phi_t d\tilde{S}_t.$$
Deduce that V_0 is the (no-arbitrage) price of the option $g(S_T) = S_T^3$.

Question 3

Let us consider the European call and put option with the same underlying asset $S = (S_t)_{0 \leq t \leq T}$ and the same strike price $K > 0$, maturity time $T > 0$. Their prices at initial time $t = 0$ are denoted by respectively $C_E(0,K)$ and $P_E(0,K)$. Assume that, for each unit of underlying asset, a dividend with amount D dollars will be issued at time $\tau_D \in (0,T)$. Prove that
$$C_E(0,K) - P_E(0,K) = S_0 - Ke^{-rT} - De^{-r\tau_D}.$$

Question 4

We observe the prices (at time $t = 0$) of the following European call/put options on the market. Suppose that the interest rate $r = 0$, and the initial price of the underlying stock is $S_0 = 100$. Please construct a portfolio, using these options together with the cash (bank account), to find an arbitrage opportunity.

Option Type	Strike	Maturity	Option Price at time $t = 0$
Put	90	2	6
Call	100	1	11
Put	110	2	14

Question 5

Let us consider two different investments, whose cash flow of each year is given as follows:

Year	0	1	2	3
Investment 1	-1000	200	500	600
Investment 2	-1000	600	500	200

Table 1: Cash flow of two investments

Assume that the (annuel) discrete compounding interest rate $r = 1\%$. Please write down the formula to compute the NPV, and then compare the two investments in terms of the NPV.