Eg: $\sin \pi z$ has zeros $0, \pm n$, $n = 1, 2, 3, \cdots$

Arrange zeros a_k s.t. $|a_{k+1}| \geq |a_k|$ & $|a_k| \to +\infty$:

$$0, \pi, -\pi, 2\pi, -2\pi, \cdots$$

order of $0 = 1$ &

i.e.

$$a_k = \begin{cases} n & \text{if } k = 2n-1, \ n = 1, 2, \cdots \\ -n & \text{if } k = 2n, \ n = 1, 2, \cdots \end{cases}$$

Infinite product

- Weierstrass:

$$z \prod_{k=1}^{\infty} E_k\left(\frac{z}{a_k}\right)$$

$$= z \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{z + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{a_k}\right)^k}$$

By (the absolute) convergence proved in the Thm, we can group consecutive terms for $k = 2n-1$ & $k = 2n$ for the same n:

$$\left(1 - \frac{z}{n}\right) e^{\left(\frac{z}{n}\right) + \frac{1}{2}\left(\frac{z}{n}\right)^2 + \cdots + \frac{1}{2n-1}\left(\frac{z}{n}\right)^{2n-1}} \left(1 - \frac{z}{(-n)}\right) e^{\left(-\frac{z}{n}\right) + \frac{1}{2}\left(-\frac{z}{n}\right)^2 + \cdots + \frac{1}{2n}\left(\frac{-z}{n}\right)^{2n}}$$

$$= \left(1 - \frac{z^2}{n^2}\right) e^{\left(\frac{z}{n}\right)^2 + \frac{1}{2}\left(\frac{z}{n}\right)^4 + \cdots + \frac{1}{n}\left(\frac{z}{n}\right)^{2n}}$$

Hence

$$\sin \pi z = e^{f(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) e^{\frac{z^2}{n^2} + \frac{1}{2}\left(\frac{z}{n}\right)^4 + \cdots + \frac{1}{n}\left(\frac{z}{n}\right)^{2n}}$$

for some entire function $f(z)$.

It is not easy to find f, but the about clearly show that factorization into infinite product may not be unique without further condition.

- Hadamard $\Rightarrow$ (provided you've proved that $\rho_{\sin\pi z} = 1$)

$$\sin \pi z = e^{a+bz} \, z \prod_{k=1}^{\infty} E_1\left(\frac{z}{a_k}\right)$$

(note that the E is always E_1, unlike Weierstrass)

$$= e^{a+bz} \, z \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k}}$$

By the (absolute) convergence proved in the Thm, we can group consecutive terms $k = 2n-1$ & $k = 2n$ for the same n:

$$\left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \left(1 - \frac{z}{(-n)}\right) e^{\left(-\frac{z}{n}\right)} = \left(1 - \frac{z^2}{n^2}\right)$$

$\therefore$ Hadamard implies $\sin \pi z = e^{a+bz} \, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

Hence, strictly speaking, this is not Hadamard factorization, but follows from Hadamard factorization.

Then one can uses properties of $\sin \pi z$ and the explicit form of the infinite product to

show that $\quad e^{a+bz} \equiv \pi \quad$ (ex!)

(for example, using
- $\lim_{z\to 0} \frac{\sin \pi z}{z}$
- $\sin \pi(-z) = -\sin \pi z$ and
- the simplified infinite product is even in z.)

- Same for $\cos z$. Even one has a factorization into infinite product (with some $e^{f(z)}$), one still needs to check it is coming from the Hadamard factorization as in the thm.

- Also note that even for Hadamard factorization

$$\sin \pi z = e^{a+bz} z \prod_{k=2}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k}},$$

one cannot write the infinite product into

$$\prod_{k=2}^{\infty} \left(1 - \frac{z}{a_k}\right) \prod_{k=2}^{\infty} e^{\frac{z}{a_k}}$$

because individual products may not converge.

- Finally, cancelling infinitely many terms of 2 infinite products needs justication

(same as subtracting infinitely many terms in a infinite series)