Lemma 2.2 Let G, f & g as in Lemma 2.1. Then

$$D(w_0,1)\setminus g(G)\neq\phi \quad \forall\, w_0\in\mathbb{C}.$$

Pf: Claim $\forall\, n\geq 1$, & $m\in\mathbb{Z}$,

$$\pm\log(\sqrt{n}+\sqrt{n-1})+\tfrac{1}{2}im\pi \notin g(G).$$

Suppose not, let $g(z)=\pm\log(\sqrt{n}+\sqrt{n-1})+\tfrac{1}{2}im\pi$ for some $n\geq 1$ & $m\in\mathbb{Z}$. (and some $z\in G$)

Then

$$2\cosh(2g(z))=e^{2g(z)}+e^{-2g(z)}$$

$$=e^{\pm 2\log(\sqrt{n}+\sqrt{n-1})}e^{im\pi}+e^{\mp 2\log(\sqrt{n}+\sqrt{n-1})}e^{-im\pi}$$

$$=e^{im\pi}\left[e^{\pm 2\log(\sqrt{n}+\sqrt{n-1})}+e^{\mp 2\log(\sqrt{n}+\sqrt{n-1})}\right]$$

$$=(-1)^m\left[(\sqrt{n}+\sqrt{n-1})^2+\frac{1}{(\sqrt{n}+\sqrt{n-1})^2}\right]$$

$$=(-1)^m\left[n+2\sqrt{n}\sqrt{n-1}+n-1+n-2\sqrt{n}\sqrt{n-1}+n-1\right]$$

$$=(-1)^m\,2(2n-1)$$

$$\Rightarrow\quad f(z)=-e^{i\pi(-1)^m(2n-1)}=-(-1)=1$$

which is a contradiction as $1\notin f(G)$

Now consider rectangle with vertexes:

$\log(\sqrt{n}+\sqrt{n-1})+\tfrac{1}{2}i(m+1)\pi$ $\qquad$ $\log(\sqrt{n+1}+\sqrt{n})+\tfrac{1}{2}i(m+1)\pi$

$\log(\sqrt{n}+\sqrt{n-1})+\tfrac{1}{2}im\pi$ $\qquad$ $\log(\sqrt{n+1}+\sqrt{n})+\tfrac{1}{2}im\pi$

(reverse horizontally if "$-$" instead of "$+$")

height of rectangle $= \frac{\pi}{2} < \sqrt{3}$

width of rectangle $= \log(\sqrt{n+1}+\sqrt{n}) - \log(\sqrt{n}+\sqrt{n-1})$

Note that $\frac{d}{dx}\left[\log(\sqrt{x+1}+\sqrt{x}) - \log(\sqrt{x}+\sqrt{x-1})\right] < 0$

width of rectangle $\leq \log(\sqrt{1+1}+\sqrt{1}) - \log(\sqrt{1}+\sqrt{1-1})$

$= \log(\sqrt{2}+1) < 1.$

Hence diagonal of the rectangle $< \left[(\sqrt{3})^2+1\right]^{\frac{1}{2}} = 2$

Therefore, for any $w_0 \in \mathbb{C}$, $D(w_0, 1)$ must contains a point in $\{\pm\log(\sqrt{n}+\sqrt{n-1}) + \frac{1}{2}im\pi : n \geq 1, m \in \mathbb{Z}\}$

and hence $D(w_0, 1) \setminus g(G) \neq \emptyset$ #

Thm 2.3 Little Picard Thm

If f is an entire function that omits two values, then f is a constant.

Pf: If $f(z) \neq a$ & $f(z) \neq b$, then $h(z) = \frac{f(z)-a}{b-a} \neq 0, 1.$

So we only need consider the case that $\{a, b\} = \{0, 1\}$.

Since $\mathbb{C}$ is simply-connected, Lemma 2.2 $\Rightarrow$

$\forall w_0 \in \mathbb{C}$, $D(w_0, 1) \setminus g(\mathbb{C}) \neq \emptyset$. (as in Lemma 2.2)

Suppose on the contrary that $g \not\equiv$ constant.

Then $g'(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$.

We may assume $z_0 = 0$.

Otherwise, consider $g(z+z_0)$ instead.

Cor 1.11 (taking $R = \frac{1}{|g'(0)|L}$) $\Rightarrow \exists\, w_0 \in \mathbb{C}$ such that

$$D(w_0, 1) \subset g\left(D\left(0, \frac{1}{|g'(0)|L}\right)\right) \subset g(\mathbb{C})$$

which is a contradiction

$\therefore\ g \equiv$ constant. ※

<u>Great Picard Thm</u> Suppose $z_0 \in \Omega$, f holo on $\Omega \setminus \{z_0\}$ and z_0 is an essential singularity of f. Then in each neighborhood of z_0, f assumes each complex number, with one possible exception, an infinite number of times

<u>Pf</u>: Omitted

Review

Ch1 Preliminaries to Cpx Analysis

Ch2 Cauchy's Thm & Its applications (§5.5 omitted)

- Holomorphic functions defined in term of integrals
  $$\int_a^b F(z,s)\,ds$$
- Schwarz reflection principle

Ch3 Meromorphic Functions & the Logarithm

Ch4 Fourier Transform

- Class $\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$
- Estimate of $\hat{f}$ for $f \in \mathfrak{F}$
- Fourier Inversion Formula (for $f \in \mathfrak{F}$)
- Poisson Summation Formula
  $$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \quad (\text{for } f \in \mathfrak{F})$$
- Theta function
- Phragmén-Lindelöf Thm (max. principle for unbdd domain)
  (other parts of §3 omitted)

Ch5 Entire Function

- Jensen's Formula
- Functions of Finite Order
  $$\rho_f = \inf\{\rho : |f(z)| \leq Ae^{B|z|^\rho}, \text{ for some } A \,\&\, B\}$$
- Weierstrass Infinite Products &
- Hadamard's Factorization Theorem (for f with $\rho_f < +\infty$)

(mid-term)

Ch6 Gamma & Zeta Functions $\Gamma(s)$ & $\zeta(s)$

- Analytic continuations of Gamma & Zeta Functions
- Various properties, formulae, and estimates for $\Gamma(s)$ & $\zeta(s)$

Ch7 Zeta Functions and Prime Number Theorem

- $\pi(x) \sim x/\log x$ as $x \to \infty$

Ch8 Conformal Mappings

- Conformal maps & conformal equivalence
- Angle preserving property
- Explicit conformal map between $\mathbb{D}$ and $\mathbb{H}$
- Fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$
  (translations, rotations, scalings, and inversion),
  maps "lines & circles" to "lines & circles"
- Elementary examples of conformal maps between specific domains.
- Dirichlet problem
- Schwarz Lemma
- Automorphism groups
  $\mathrm{Aut}(\mathbb{D})$, $\mathrm{Aut}(\mathbb{H})$ (and $\mathrm{Aut}_0(\mathbb{D})$)
- Riemann Mapping Theorem
- Normal Family and Montel's Theorem
- Hurwitz Thm (and corresponding Prop 3.5)

- Conformal Maps onto Polygons,
- Carathéodory extension to the boundary
- Schwarz-Christoffel Integral, Elliptic Integral

Ch9 <u>Elliptic Functions</u>

- Fundamental parallelogram, period parallelograms
- Weierstrass $\wp$ function
- Modular Character
- Eisenstein Series

<u>Final exam</u>: $\boxed{\text{May 6 (Tuesday) 3:30-5:30 pm, UC gym.}}$

covers all material including those in lectures, tutorials, homework, & textbook (including all exercises in Textbook no matter it's assigned in homework or not) up to ch9, except Ch7, with emphasis on those material after the mid-term.

But those material before mid-term may also be tested directly/explicitly or indirectly/implicitly.

Approved calculator allowed.