2. Modular Character of Elliptic Functions & Eisenstein Series

Note: $\wp$ depends on $\tau \in \mathbb{H}$ and will be denoted by $\wp_\tau$ in this section as we will study how $\wp_\tau$ depends on τ. (Modular Character of $\wp$)

Easy Facts:

$$\boxed{\begin{aligned} &(1)\quad \wp_\tau(z) = \wp_{\tau+1}(z) \\ &(2)\quad \wp_{-\frac{1}{\tau}}(z) = \tau^2 \wp_\tau(\tau z) \qquad (\text{Note: } \tau \in \mathbb{H} \Leftrightarrow -\tfrac{1}{\tau} \in \mathbb{H}) \end{aligned}}$$

Pf of (1): $\forall n, m \in \mathbb{Z}$, $\quad n + m(\tau+1) = (n+m) + m\tau$

$$\Rightarrow \quad \Lambda_{\tau+1} = \Lambda_\tau$$

Also, $\quad (n+m, m) = (0,0) \Leftrightarrow (n,m) = (0,0)$

$$\therefore \quad \Lambda^*_{\tau+1} = \Lambda^*_\tau$$

Therefore, $\displaystyle \wp_{\tau+1}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_{\tau+1}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_\tau} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] \qquad (\text{by abs. convergence})$$

$$= \wp_\tau(z)$$

Pf of (2) $\quad \forall n, m \in \mathbb{Z} \quad n + m\left(-\frac{1}{\tau}\right) = \dfrac{-m + n\tau}{\tau}.$

$$\therefore \quad \omega \in \Lambda^*_{-\frac{1}{\tau}} \Leftrightarrow \tau\omega \in \Lambda^*_\tau$$

$$\Rightarrow \quad \wp_{-\frac{1}{\tau}}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*_{-\frac{1}{\tau}}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

$$= \frac{1}{z^2} + \sum_{\tau\omega \in \Lambda^*_\tau} \left[\frac{\tau^2}{(\tau z + \tau\omega)^2} - \frac{\tau^2}{(\tau\omega)^2} \right]$$

$$= \tau^2 \left\{ \frac{1}{(\tau z)^2} + \sum_{\omega' \in \Lambda^*_\tau} \left[\frac{1}{(\tau z + \omega')^2} - \frac{1}{\omega'^2} \right] \right\}$$

$$= \tau^2 \wp_\tau(\tau z) \quad \#$$

<u>Def</u>: <u>Modular group</u> = group of transfamations of $\mathbb{H}$ generated by $\tau \mapsto \tau+1$ & $\tau \mapsto -\frac{1}{\tau}$

<u>Notes</u>: Modular group is a subgroup of $\mathrm{Aut}(\mathbb{H})$

as $\tau + 1 = \dfrac{1\cdot\tau+1}{0\cdot\tau+1}$ & $-\dfrac{1}{\tau} = \dfrac{0\cdot\tau+(-1)}{1\cdot\tau+0}$

2.1 Eisenstein Series

<u>Def</u> The <u>Eisenstein series</u> of <u>order k</u> is defined by, $\forall \tau \in \mathbb{H}$,

$$E_k(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^k} = \sum_{\omega\in\Lambda^*_\tau} \frac{1}{\omega^k}$$

where $\Lambda^*_\tau = \Lambda_\tau \setminus \{(0,0)\}$ & Λ_τ = lattice generated by 1 & τ

Thm 2.1 (i) If $k \geq 3$, $E_k(\tau)$ converges & is holo. in $\mathbb{H}$.

(ii) If $k = $ odd, $E_k(\tau) = 0$.

(iii) $E_k(\tau+1) = E_k(\tau)$ & $E_k(\tau) = \frac{1}{\tau^k} E_k(\frac{-1}{\tau})$ (modular character)

Pf: (i) If $k \geq 3$, Lemma 1.5 & its proof

$\Rightarrow$ $E_k(\tau)$ converges abolutely and uniformly on compact subsets of $\mathbb{H}$.

$\therefore$ $E_k(\tau)$ converges to a holo. on $\mathbb{H}$

(ii) Since $\omega \in \Lambda_\tau^* \Leftrightarrow -\omega \in \Lambda_\tau^*$,

$$E_k(\tau) = \sum_{\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = \sum_{-\omega \in \Lambda_\tau^*} \frac{1}{(-\omega)^k} = (-1)^k \sum_{-\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = (-1)^k E_k(\tau)$$

$\therefore$ $k = $ odd $\Rightarrow$ $E_k(\tau) = 0$

(iii) Using $\Lambda_{\tau+1}^* = \Lambda_\tau^*$, we have

$$E_k(\tau+1) = \sum_{\omega \in \Lambda_{\tau+1}^*} \frac{1}{\omega^k} = \sum_{\omega \in \Lambda_\tau^*} \frac{1}{\omega^k} = E_k(\tau)$$

Using $\omega \in \Lambda_{-\frac{1}{\tau}}^* \Leftrightarrow \tau\omega \in \Lambda_\tau^*$, we have

$$E_k(-\tfrac{1}{\tau}) = \sum_{\omega \in \Lambda_{-\frac{1}{\tau}}^*} \frac{1}{\omega^k} = \sum_{\tau\omega \in \Lambda_\tau^*} \frac{\tau^k}{(\tau\omega)^k} = \tau^k \sum_{\omega' \in \Lambda_\tau^*} \frac{1}{(\omega')^k} = \tau^k E_k(\tau)$$

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$\boxed{\text{Thm 2.2 } \forall \tau \in \mathbb{H}, \quad \wp_\tau(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2}(\tau) z^{2k} \quad \text{near } z=0}$

Pf: For simplicity, τ will be omitted in the following calculation.

By definition,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

$$= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \qquad (\text{since } \omega\in\Lambda^* \Leftrightarrow -\omega\in\Lambda^*)$$

$$= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\frac{1}{\omega^2}\left[\frac{1}{(1-\frac{z}{\omega})^2} - 1\right]$$

near $z=0$ $$= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\frac{1}{\omega^2}\left[\sum_{l=0}^{\infty}(l+1)\left(\frac{z}{\omega}\right)^l - 1\right]$$

$$\left(\frac{1}{1-\xi} = \sum_{l=0}^{\infty}\xi^l \Rightarrow \frac{1}{(1-\xi)^2} = \sum_{l=0}^{\infty} l\xi^{l-1}\right)$$

$\therefore$ near $z=0$

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\sum_{l=1}^{\infty}(l+1)\frac{z^l}{\omega^{l+2}}$$

$$= \frac{1}{z^2} + \sum_{l=1}^{\infty}(l+1)\left(\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{l+2}}\right)z^l \qquad \text{by absolute convergence}$$

$$= \frac{1}{z^2} + \sum_{l=1}^{\infty}(l+1)E_{l+2}z^l$$

By Thm 2.1, $E_{odd} = 0$, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}z^{2k}, \text{ near } z=0 .$$

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