<u>Prop 4.1</u> Suppose $S(z)$ is given by (5) in the above definition and $a_1, \cdots, a_n$ & a_∞ are as in the remarks (iii) & (iv).

(i) If

- $\sum_{k=1}^{n} \beta_k = 2$, and
- $\mathfrak{p}$ denotes the polygon whose vertices are given by $a_1, \cdots, a_n$ (in order)

("polygon" = a closed curve consists of finitely many line segments.)

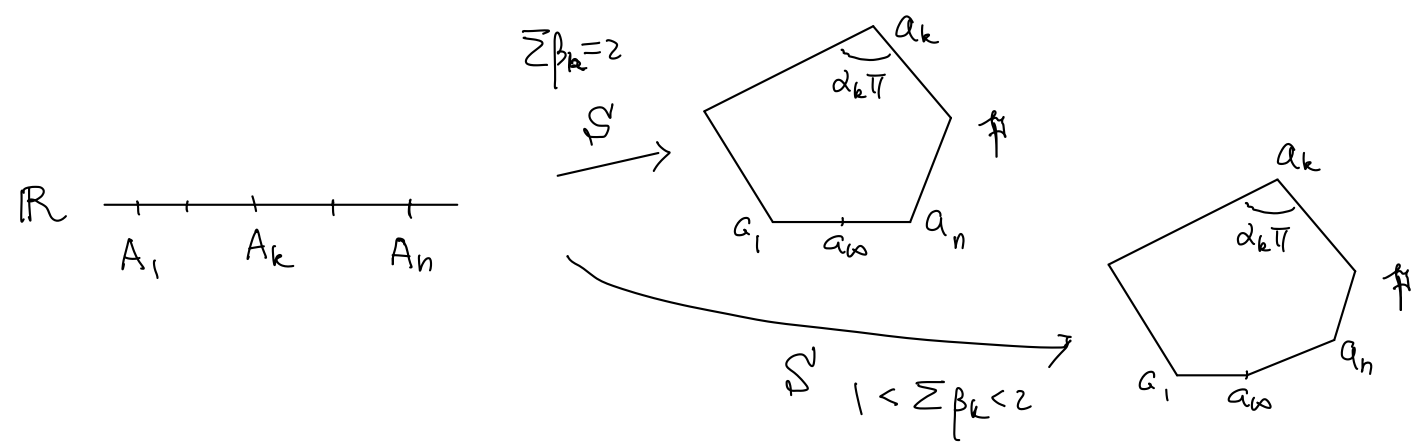
then

- $a_\infty = S(\infty)$ lies on the segment $[a_n, a_1]$
- $S(\mathbb{R}) = \mathfrak{p} \setminus \{a_\infty\}$
- (Interior) angle at $a_k = \alpha_k \pi$, $\alpha_k = 1 - \beta_k$.

(ii) If $1 < \sum_{k=1}^{n} \beta_k < 2$, the similar conclusion holds with

- $\mathfrak{p}$ replaced by the polygon of $n+1$ sides with vertices $a_1, a_2, \cdots a_n, a_\infty$ (in order), and
- (Interior) angle at $a_\infty = \alpha_\infty \pi$,

$$\alpha_\infty = 1 - \beta_\infty \quad \& \quad \beta_\infty = 2 - \sum_{k=1}^{n} \beta_k .$$



Pf Case (i) $\sum_{k=1}^{n} \beta_k = 2$

If $A_k < x < A_{k+1}$, $k = 1, \cdots, n-1$.

Then $$S'(x) = \frac{1}{\left[(x-A_1)^{\beta_1} \cdots (x-A_k)^{\beta_k}\right]\left[(x-A_{k+1})^{\beta_{k+1}} \cdots (x-A_n)^{\beta_n}\right]}$$

By the choice of branch of each $x - A_j$ in Remark (ii),

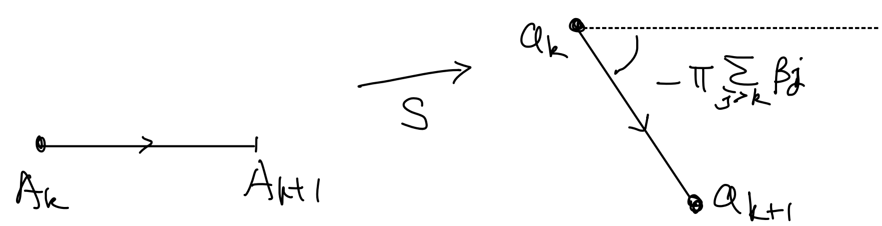
$$\arg (x-A_j)^{\beta_j} = \begin{cases} 0 & \text{for } j \leq k \\ \pi \beta_j & \text{for } j > k \end{cases}$$

$\therefore$ $\arg S'(x) = -\pi \sum_{j>k} \beta_j$ which is a constant for $x \in (A_k, A_{k+1})$.

$\Rightarrow$ $S[A_k, A_{k+1}]$ is a straight line segment that makes an angle of $-\pi \sum_{j>k} \beta_j$ with the x-axis.

Notice that $S(x) = S(A_k) + \int_{A_k}^{x} S'(y)\,dy \quad \forall x \in (A_k, A_{k+1})$.

$S(x)$ varies from end point $a_k = S(A_k)$ to end point $a_{k+1} = S(A_{k+1})$ as x varies from A_k to A_{k+1}.



Similarly $$\arg S'(x) = \begin{cases} 0 & \text{if } x > A_n \quad (\text{i.e. } S'(x) > 0) \\ -\pi \sum_{k=1}^{n} \beta_k = -2\pi, & \text{if } x < A_1 \end{cases}$$

And • $S(x)$ varies from $a_n = S(A_n)$ to $a_\infty = S(A_\infty)$

as x varies from A_n to ∞.

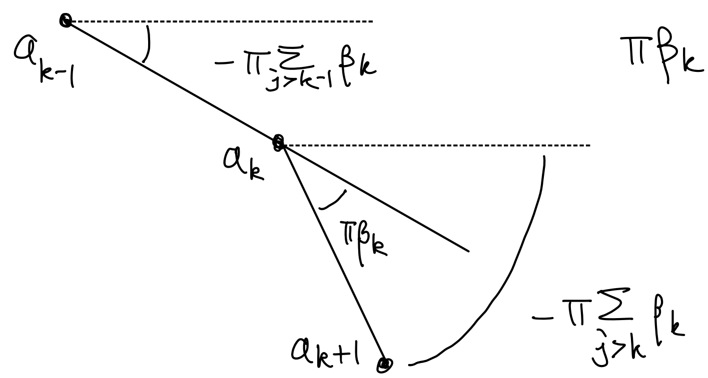
• $S(x)$ varies from a_∞ to $a_1 = S(A_1)$

as x varies from $-\infty$ to A_1

This shows that $a_\infty \in [a_1, a_n]$ (angles with x-axis $= 0$ & -2π)

This proves $S(\mathbb{R}) = P \setminus \{a_\infty\}$.

Note that

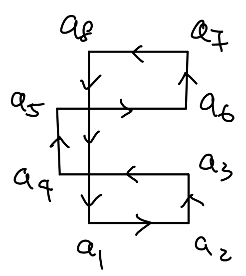


$$\pi\beta_k = \left(-\pi \sum_{j>k} \beta_k\right) - \left(-\pi \sum_{j>k-1} \beta_k\right)$$

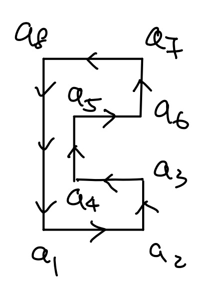
$\therefore$ Interior angle at $a_k = \pi - (\pi\beta_k) = \alpha_k \pi$.

Case (ii) $1 < \sum_{k=1}^{n} \beta_k < 2$ is similar (Ex!) ##

Notes: (i) For an arbitrary choice of n; $A_1, \cdots, A_n$; $\beta_1, \cdots, \beta_n$, the "polygon" P in Prop 4.1 may <u>not</u> be <u>simple</u>. The following could happen:



(compares



)

(ii) Even $\mathfrak{P} = \partial P$, P simply-connected region, Prop 4.1 hasn't shown that $S : \mathbb{H} \to P$ is conformal. (See subsection 4.4 below)