Prop 3.5 Let
- $\Omega \subset \mathbb{C}$ be a region, &
- $\{f_n\}$, f be holo functions on Ω such that
- $f_n \to f$ uniformly on every compact subset of Ω

If f_n are injective, then

f is either injective or constant.

Pf: Suppose that f is not injective.

Then $\exists\ z_1, z_2 \in \Omega$ such that

$$z_1 \neq z_2 \text{ but } f(z_1) = f(z_2).$$

Define $g_n(z) = f_n(z) - f_n(z_1)$.

Then $\begin{cases} g_n(z_1) = 0 \ \& \\ g_n(z) \neq 0, \ \forall z \in \Omega \setminus \{z_1\}, \end{cases}$ since f_n injectives

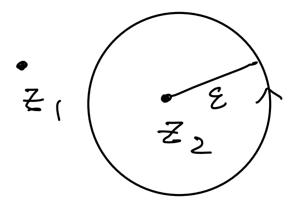
As $f_n \to f$ uniformly on cpt subset,

$$g_n \to g = f - f(z_1) \text{ uniformly on cpt. subset.}$$

If $g \not\equiv 0$, then z_2 is an isolated zero of g.

$(g(z_2) = f(z_2) - f(z_1) = 0)$

$$\Rightarrow \quad 1 \leq \frac{1}{2\pi i} \int_{|\zeta - z_2| = \varepsilon} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta$$



along a small circle $|\zeta - z_2| = \varepsilon$ around z_2

s.t. $g(\zeta) \neq 0$, $\forall\ |\zeta - z_2| \leq \varepsilon$.

Then $\frac{1}{g_n} \to \frac{1}{g}$ uniformly on $|\zeta - z_2| = \varepsilon$

and hence $\displaystyle \frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon} \frac{g_n'(\zeta)}{g_n(\zeta)}\,d\zeta \longrightarrow \frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon} \frac{g'(\zeta)}{g(\zeta)}\,d\zeta \geq 1$

This is a contradiction as g_n has no zero in $|\zeta - z_2| \leq \varepsilon$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_{|\zeta-z_2|=\varepsilon} \frac{g_n'(\zeta)}{g_n(\zeta)}\,d\zeta = 0, \qquad \forall n,$$

$\therefore \quad g \equiv 0 \quad \Rightarrow \quad f(z) = f(z_1)$ a constant $\forall z \in \Omega$

$\divideontimes$

<u>Remark</u>: The argument in the proof of Prop 3.5 gives the following

<u>Hurwitz Theorem</u>:

If f_n & f analytic in Ω, $f_n(z) \neq 0$, $\forall z \in \Omega$, and f_n converges uniformly to f on every compact set of Ω,

then either (i) $f(z) \equiv 0$, or

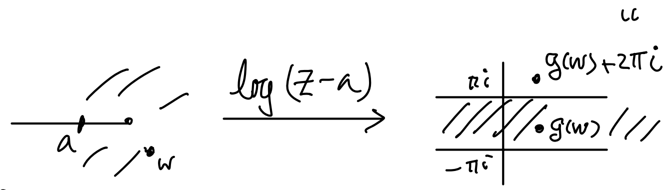
(ii) $f(z) \neq 0$, $\forall z \in \Omega$.

And clearly Hurwitz Thm $\Rightarrow$ Prop 3.5.

3.3 Proof of the Riemann Mapping Theorem

> **Step 1** For a proper and simply-connected region Ω, and $z_0 \in \Omega$, $\exists$ conformal
>
> $$f : \Omega \to f(\Omega) \subset \mathbb{D} \text{ s.t. } f(z_0) = 0 \ \& \ f'(z_0) > 0$$

Pf: Ω is proper $\Rightarrow \exists\, a \in \mathbb{C} \setminus \Omega$.



Then Ω is simply-connected $\Rightarrow$

$g(z) = \log(z-a)$ is well-defined in Ω.

Clearly (i) g is injective

(ii) $\forall\, w \in \Omega$, $g(z) \neq g(w) + 2\pi i$, $\forall\, z \in \Omega$

(by taking exponential)

Claim: $\forall\, w \in \Omega$, $\exists\, r > 0$ s.t. $\overline{D_r(g(w) + 2\pi i)} \cap g(\Omega) = \emptyset$.

Pf: Suppose not, then $\exists\, z_n \in \Omega$ such that

$$g(z_n) \to g(w) + 2\pi i.$$

Taking exponential, $z_n \to w$.

And hence $g(z_n) \to g(w)$ which is a contradiction.

Then $h(z) = \dfrac{r}{g(z) - (g(w) + 2\pi i)}$ is holo. injective and $|h(z)| = \dfrac{r}{|g(z) - (g(w)+2\pi i)|} < \dfrac{r}{r} = 1$

$\therefore \quad h : \Omega \to h(\Omega) \subset \mathbb{D}$ conformal

Finally, $f(z) = e^{i\theta} \dfrac{h(z_0) - h(z)}{1 - \overline{h(z_0)}\, h(z)} = e^{i\theta}\, \psi_{h(z_0)} \circ h(z)$

(where ψ_α as in subsection 2.1 & $\theta \in \mathbb{R}$ to be chosen)

is holo. injective, $f(\Omega) \subset \mathbb{D}$, and $f(z_0) = 0$.

Furthermore $f'(z_0) = e^{i\theta}\, \psi'_{h(z_0)}(h(z_0))\, h'(z_0)$.

Hence if $\theta = -\arg\left(\psi'_{h(z_0)}(h(z_0))\, h'(z_0)\right)$,

then $f'(z_0) > 0$. $\#$