## Properties of fractional linear transformations

(1) Conformal as maps from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$, hence angles preserving.

(2) $f, g$ are fractional linear transformations

$\Rightarrow$ $f\circ g$ is a fractional linear transformation.

(3) fractional linear transformation is a composition of translations, dilations and inversions.

(4) fractional linear transformations map "straight lines & circles" to "straight lines or circles".

Pf: (1) Clearly $f(z)=\frac{az+b}{cz+d}$ has derivatives

$$f'(z)=\frac{ad-bc}{(cz+d)^2} \neq 0 \quad \text{for } z \neq -\frac{d}{c}.$$

(we omit the discussion at $z=-\frac{d}{c}$ and $z=\infty$)

Also, clearly $g(w)=\dfrac{dw-b}{-cw+a}$ is the inverse of $f$

(Note: $z=-\frac{d}{c} \leftrightarrow w=\infty$, $z=\infty \leftrightarrow w=\frac{a}{c}$)

$\therefore$ $f$ is conformal (from $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(2) If $f(z)=\dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

$g(z)=\dfrac{kz+l}{mz+n}$, $kn-lm \neq 0$

Then $f\circ g(z) = \dfrac{a\left(\frac{kz+l}{mz+n}\right)+b}{c\left(\frac{kz+l}{mz+n}\right)+d} = \dfrac{(ak+bm)z+(al+bn)}{(ck+dm)z+(cl+dn)}$

Note that $\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}$

$\therefore$ $(ak+bm)(cl+dn) - (al+bm)(ck+dm)$

$$= \det\begin{pmatrix} ak+bm & al+bn \\ ck+dm & cl+dn \end{pmatrix}$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} k & l \\ m & n \end{pmatrix}$$

$$= (ad-bc)(kn-lm) \neq 0 .$$

$\therefore$ $f\circ g$ is a fractional linear transformation.

(3) $f(z) = \dfrac{az+b}{cz+d}$ , $ad-bc \neq 0$

If $c=0$, then $d \neq 0$ & $f(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$

i.e. $z \mapsto \underset{\substack{\uparrow \\ \text{dilation } (a\neq 0)}}{\left(\frac{a}{d}\right)z} \longmapsto \left[\left(\frac{a}{d}\right)z\right] \underset{\substack{\uparrow \\ \text{translation}}}{+} \left(\frac{b}{d}\right) = f(z)$

If $c \neq 0$, then

$$f(z) = \frac{az+b}{cz+d} = \frac{1}{c} \cdot \frac{az+b}{z+\frac{d}{c}}$$

$$= \frac{1}{c}\left[\frac{a(z+\frac{d}{c}) - \frac{ad}{c} + b}{z+\frac{d}{c}}\right]$$

$$= \frac{1}{c}\left[a - \frac{\frac{ad}{c} - b}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c} - \frac{(ad-bc)}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$$

i.e. $\underset{\substack{\uparrow \\ \text{translation}}}{z \mapsto z+\frac{d}{c}} \underset{\substack{\uparrow \\ \text{inversion}}}{\to} \frac{1}{z+\frac{d}{c}} \underset{\substack{\uparrow \\ \text{dilation}}}{\mapsto} -\frac{(ad-bc)}{c^2}\frac{1}{z+\frac{d}{c}}$

$\underset{\substack{\uparrow \\ \text{translation.}}}{\mapsto} \frac{a}{c} - \frac{ad-bc}{c^2}\frac{1}{z+\frac{d}{c}}$

(4) Note that translations and dilations map straight lines to straight lines, and circles to circles.

Then because of (3), we only need to prove (4) for inversion $z \mapsto \frac{1}{z}$.

let $z = x+iy$ & $w = s+it = \frac{1}{z}$

then $s+it = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

i.e. $\begin{cases} s = \dfrac{x}{x^2+y^2} \\ t = -\dfrac{y}{x^2+y^2} \end{cases}$ (Inversion as a mapping from $\mathbb{R}^2\setminus\{0\} \to \mathbb{R}^2\setminus\{0\}$)

Also $wz = 1 \Rightarrow |w|^2|z|^2 = 1$,

(ie. $s^2+t^2 = \frac{1}{x^2+y^2}$) $\Rightarrow \begin{cases} x = \frac{s}{s^2+t^2} \\ y = \frac{-t}{s^2+t^2} \end{cases}$

Now let $L: ax+by+c=0$ be a straight line

Then $$\frac{as}{s^2+t^2} - \frac{bt}{s^2+t^2} + c = 0$$

ie. $$c(s^2+t^2) + as - bt = 0$$

If $c=0$ (ie. $L$ passing thro the origin),

the image of $L$ is the straight line

$$L' : as - bt = 0 \quad (\text{in } (s,t)\text{-plane}).$$

If $c \neq 0$ (ie. $L$ not passing thro the origin)

$\therefore$ the image of $L$ is the circle

$$C' : s^2+t^2+\left(\frac{a}{c}\right)s - \frac{b}{c}t = 0 \quad (\text{in } (s,t)\text{-plane})$$

Now let $C: x^2+y^2+ax+by+c=0$ be a circle.

Then we have $$\frac{1}{s^2+t^2} + \frac{as}{s^2+t^2} - \frac{bt}{s^2+t^2} + c = 0$$

$$\Rightarrow c(s^2+t^2) + as - bt + 1 = 0.$$

If $c=0$, the image of $C$ is a straight line

$$L' : as - bt + 1 = 0$$

If $c \neq 0$, the image of $C$ is a circle

$$C' : s^2+t^2+\left(\frac{a}{c}\right)s - \left(\frac{b}{c}\right)t + \frac{1}{c} = 0.$$

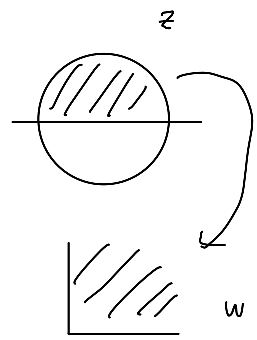
$\#$

Eg3 (of the Text book)

$$f(z) = \frac{1+z}{1-z} : \{ z = x+iy : |z| < 1 \text{ and } y > 0 \} = \mathbb{D}^+$$

$$\to \{ w = u+iv : u > 0 \text{ and } v > 0 \} = S$$

is conformal.

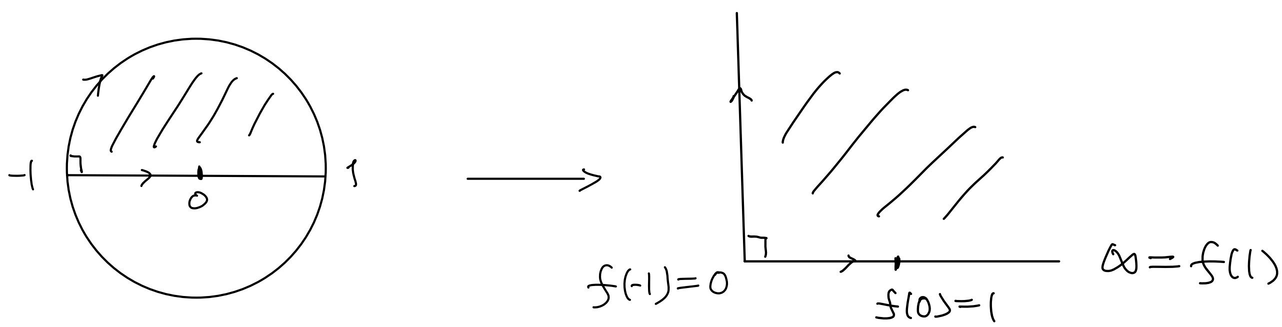



Note: $f$ is a fractional linear transformation

$$f(z) = \frac{z+1}{-z+1} \quad \text{with} \quad 1 \cdot 1 - (1)(-1) = 2 \neq 0$$

$\therefore$ $f$ is injective, hence remain to show $f(\mathbb{D}^+) = S$.

Observe that $f(-1) = 0$, $f(0) = 1$, $f(1) = \infty$




By property (4) of fractional linear transformation,
the real line segment between $-1$ & $1$
maps to part of a straight line or a circle.

Since it passes through $f(-1) = 0$, $f(0) = 1$ & $f(1) = \infty$,
it is the positive real axis.

Similarly, the upper semi-circle maps to part of a straight line or a circle passing through 0 and $\infty$, and hence must be a straight line.

Since the angles from $[-1,1]$ to the semi-circle is $\frac{\pi}{2}$, the angle from the positive x-axis to the image straight line of the semi-circle is also $\frac{\pi}{2}$ ($f$ conformal)

$\therefore$ the image of the upper semi-circle is the positive y-axis.

(Positivity can also be confirmed by $f(i) = \frac{1+i}{1-i} = i$)

This shows that $f(\mathbb{D}^+) = S$ (as $f$ is conformal : $\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$)

(Of course, all these can be proved by using coordinates as in the Textbook)