1.1 Estimates for $1/\zeta(s)$

Prop 1.6 $\forall \varepsilon > 0$, $\exists\, C_\varepsilon > 0$ s.t.

$$\frac{1}{|\zeta(s)|} \leq C_\varepsilon |t|^\varepsilon \quad \text{for } s = \sigma + it, \ \sigma \geq 1 \text{ and } |t| \geq 1.$$

Pf: By Cor 1.5 and $\zeta(s)$ only has a pole at $s=1$, we have

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1, \quad \forall\, \sigma \geq 1$$

By Prop 2.7 (i) of Ch6, (taking $\sigma_0 = 1$)

$$|\zeta(\sigma+2it)| \leq C_1 |t|^\varepsilon \quad \forall\, \sigma \geq 1 \ \& \ |t| \geq 1. \quad (C_1 = C_1(\varepsilon) > 0)$$

Hence

$$1 \leq |\zeta^3(\sigma)\zeta^4(\sigma+it)| \cdot C_1 |t|^\varepsilon$$

Then similar to $(*)_2$ in the proof of Thm 1.2,

$$|\zeta^3(\sigma)| \leq \frac{C_2}{(\sigma-1)^3} \quad \text{for } \sigma > 1. \quad (C_3 = C_3(\varepsilon) > 0)$$

Hence $$|\zeta^4(\sigma+it)| \geq \frac{C_3(\sigma-1)^3}{|t|^\varepsilon} \quad \forall\, \sigma > 1 \ \& \ |t| \geq 1$$

and clearly this inequality trivially holds for $\sigma = 1$.

Hence

$$(3) \quad |\zeta(\sigma+it)| \geq C_4 (\sigma-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}, \quad \forall\, \sigma \geq 1 \ \& \ |t| \geq 1$$

$$(C_4 = C_4(\varepsilon) > 0)$$

Note that by Prop 2.7 (ii) of Ch 6, we have

for $\sigma' > \sigma \geq 1$,

$$|\zeta(\sigma'+it) - \zeta(\sigma+it)| \leq |\zeta'(\sigma_c+it)||\sigma'-\sigma| \quad \text{for some } \sigma \leq \sigma_c \leq \sigma'$$

$$\leq C_5 |t|^{\varepsilon} |\sigma'-\sigma| \quad (C_5 = C_5(\varepsilon) > 0)$$

$$\leq C_5 |t|^{\varepsilon} (\sigma'-1). \quad (\sigma' > \sigma \geq 1)$$

Let $A = \left(\dfrac{C_4}{2C_5}\right)^4 > 0$.

<u>Case 1</u> $\sigma - 1 \geq A|t|^{-5\varepsilon}$

Then (3) $\Rightarrow$ $|\zeta(\sigma+it)| \geq C_4 (A|t|^{-5\varepsilon})^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}$

$$= (C_4 A^{\frac{3}{4}}) |t|^{-4\varepsilon}$$

<u>Case 2</u> $\sigma - 1 < A|t|^{-5\varepsilon}$

Take $\sigma' > \sigma$ such that $\sigma' - 1 = A|t|^{-5\varepsilon}$.

Then triangle inequality $\Rightarrow$

$$|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - |\zeta(\sigma'+it) - \zeta(\sigma+it)|$$

$$\geq C_4 (\sigma'-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon} (\sigma'-1)$$

$$= \left[C_4 (\sigma'-1)^{-\frac{1}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[C_4 \frac{1}{(A|t|^{-5\varepsilon})^{\frac{1}{4}}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[C_4 \cdot \frac{2C_5}{C_4} \cdot |t|^{\varepsilon} - C_5 |t|^{\varepsilon} \right] (\sigma' - 1)$$

$$= C_5 |t|^{\varepsilon} (\sigma' - 1)$$

$$= C_5 A \, |t|^{-4\varepsilon}$$

Hence $\forall \varepsilon > 0$, $\quad |\zeta(\sigma + it)| \geq C_\varepsilon |t|^{-4\varepsilon}$

where $\quad C_\varepsilon = \min\{ C_4 A^{\frac{3}{4}}, C_5 A \}$.

Replacing 4ε by ε, we have

$|\zeta(\sigma + it)| \geq C_\varepsilon |t|^{-\varepsilon}$ with a new C_ε. #

2. Reduction to the functions ψ and ψ_1

Def Tchebychev's ψ-function

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p$$

$$= \sum_{1 \leq n \leq x} \Lambda(n)$$

where $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \ \& \ m \geq 1 \\ 0, & \text{otherwise} \end{cases}$

Remarks: (i) The sum $\sum_{p^m \leq x}$ is over those integers of the form p^m & $\leq x$.

(ii) $[u]$ = greatest integer $\leq u$.

Prop 2.1 If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$

Pf omitted as it is completely a "real" analytic argument.
(Reading Exercise)

Remark: Converse of Prop. 2.1 holds.

Def $\psi_1(x) = \int_1^x \psi(u)\,du$

Prop 2.2 If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$, and therefore $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

Pf omitted as it is completely a "real" analytic argument.

(Reading Exercise)