Lemma 5.5 Suppose g is entire, if $\exists$ seq. $\{r_m\}$, $r_m \to +\infty$ s.t.

$$\operatorname{Re} g(z) \le C r_m^s \quad \text{for } |z| = r_m, \ \forall m \ge 1.$$

Then g is a polynomial of degree $\le s$.

Pf: g entire $\Rightarrow$ $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $\forall z \in \mathbb{C}$

By Cauchy integral formula (Fourier coefficients),

we have

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta}) e^{-in\theta} d\theta = \begin{cases} b_n r^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

$\Rightarrow$ For $n > 0$, $\frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})} e^{-in\theta} d\theta = 0$

Hence $\frac{1}{2\pi}\int_0^{2\pi} (g + \bar{g})(re^{i\theta}) e^{-in\theta} d\theta = b_n r^n$, $n > 0$

ie. $\int_0^{2\pi} [\operatorname{Re} g(re^{i\theta})] \cdot e^{-in\theta} d\theta = \pi b_n r^n$, $\forall n > 0$

For $n = 0$, $\int_0^{2\pi} \operatorname{Re} g(re^{i\theta}) d\theta = 2\pi \operatorname{Re}(b_0)$.

Note that $\int_0^{2\pi} e^{-in\theta} d\theta = 0$, $\forall n > 0$,

we have $b_n = \frac{1}{\pi r^n}\int_0^{2\pi} [\operatorname{Re} g(re^{i\theta}) - Cr^s] e^{-in\theta} d\theta$

$\Rightarrow$ for $r = r_m$,

$$|b_n| \leq \frac{1}{\pi r_m^n} \int_0^{2\pi} [C r_m^s - \operatorname{Re} g(r_m e^{i\theta})] \, d\theta$$

$$= \frac{2C}{r_m^{n-s}} - \frac{2\operatorname{Re}(b_0)}{r_m^n} \longrightarrow 0 \quad \text{as } r_m \to +\infty \quad \text{if } n > s$$

$\therefore$ g = poly. of degree $\leq s$. #

Pf of Hadamard's Theorem

Let $E(z) = z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$

Lemma 4.2 $\Rightarrow$ $\left|1 - E_k\left(\frac{z}{a_n}\right)\right| \leq c \left|\frac{z}{a_n}\right|^{k+1}$ for some c

Thm 2.1 $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < +\infty$ since $k+1 > \rho_f$

we have $\sum_{n=1}^{\infty} \left|1 - E_k\left(\frac{z}{a_n}\right)\right| \leq C|z|^{k+1}$ $C > 0$ indep. of z.

Hence the infinite product converges uniformly on $\{|z| \leq R\}$, $\forall R > 0$,

this implies $E(z)$ is a well-defined entire function.

Since $E(z)$ has the same zeros as $f(z)$,

$\frac{f(z)}{E(z)}$ is holomorphic and nowhere vanishing.

$\Rightarrow$ $\frac{f(z)}{E(z)} = e^{g(z)}$ for some entire $g(z)$.

By Cor 5.4. for $|z| = r_m$,

$$e^{\operatorname{Re} g(z)} = \left|\frac{f(z)}{E(z)}\right| \leq \frac{Ae^{B|z|^s}}{e^{-C|z|^s}} \qquad \forall\ s > \rho_f$$

$$= Ae^{(B+C)|z|^s}$$

$\Rightarrow \forall\ |z| = r_m$, $\quad \operatorname{Re} g(z) \leq C|z|^s$. (diff. C.)

By Lemma 5.5, $g(z)$ = polynomial of degree $\leq s$.

$\Rightarrow$ $\quad g(z)$ = polynomial of degree $\leq k$.

#

Ch6 The Gamma and Zeta Functions

§1 The Gamma Function

<u>Def</u>: For $s>0$, the <u>gamma function</u> is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\, dt. \qquad \text{———} \quad (1)$$

<u>Remark</u>: By definition,

$$\int_0^{\infty} e^{-t} t^{s-1}\, dt = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t} t^{s-1}\, dt$$

Clearly $\Gamma(s)$ is well-defined for $s>0$:

$$\int_{\varepsilon}^{1} t^{s-1}\, dt = \left.\frac{t^s}{s}\right|_{\varepsilon}^{1} \to \frac{1}{s} \quad \text{as } \varepsilon \to 0.$$

(and e^{-t} bdd near $t=0$ & rapidly decay as $t \to +\infty$)

<u>Prop 1.1</u> The formula $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1}\, dt$ extends the domain of definition of $\Gamma(s)$ to the open half-plane

$\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$.

<u>Pf</u>: It suffices to show that (1) defines a holomorphic function in $\quad S_{\delta,M} = \{\delta < \mathrm{Re}(s) < M\}, \quad \forall\, 0 < \delta < M < \infty.$

Note that $t^{s-1} = e^{(s-1)\log t}$ is holomorphic in $s \in \mathbb{C}$

and hence $e^{-t} t^{s-1}$ is holo. in s, and continuous

in $(t, s) \in [\varepsilon, \frac{1}{\varepsilon}] \times [\delta, M]$.

Hence Thm 5.4 of Ch2 $\Rightarrow$

$\forall \varepsilon > 0$, $\quad F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt \quad$ is holo. on $S_{\delta, M}$.

Note also that $\delta < Re(s) < M$

$$\Rightarrow \quad |e^{-t} t^{s-1}| = e^{-t} |e^{(s-1)\log t}| = e^{-t} e^{(Re(s)-1)\log t}$$
$$= e^{-t} t^{Re(s)-1}$$

(for $\varepsilon < 1$)
$$\Rightarrow \int_0^\varepsilon |e^{-t} t^{s-1}| dt = \int_0^\varepsilon e^{-t} t^{Re(s)-1} dt$$
$$\le \int_0^\varepsilon e^{-t} t^{\delta - 1} dt \le \frac{\varepsilon^\delta}{\delta}$$

$\therefore \int_0^\varepsilon e^{-t} t^{s-1} dt$ is convergent. (as improper integral near 0)

Similarly $\displaystyle \int_{\frac{1}{\varepsilon}}^\infty |e^{-t} t^{s-1}| dt \le \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{M-1} dt$

$$\le C \int_{\frac{1}{\varepsilon}}^\infty e^{-\frac{t}{2}} dt \quad (\text{for some } C > 0)$$

$$\le 2C e^{-\frac{1}{2\varepsilon}}$$

$\therefore \quad \int_{\frac{1}{\varepsilon}}^{\infty} e^{-t} t^{s-1}\,dt$ is also convergent.

Hence $\int_0^\infty e^{-t} t^{s-1}\,dt$ defined &

$$\int_0^\infty e^{-t} t^{s-1}\,dt - F_\varepsilon(s) = \int_0^\varepsilon e^{-t} t^{s-1}\,dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{s-1}\,dt$$

$$\Rightarrow \left|\int_0^\infty e^{-t} t^{s-1}\,dt - F_\varepsilon(s)\right| \le \int_0^\varepsilon e^{-t} t^{Re(s)-1}\,dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t} t^{Re(s)-1}\,dt$$

$$\le \frac{\varepsilon^\delta}{\delta} + 2Ce^{-\frac{1}{2\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0.$$

$\therefore$ (Thm 5.2 of Ch2) $\int_0^\infty e^{-t} t^{s-1}\,dt$ is a uniform limit of sequence of holo. functions on $S_{\delta,M}$, hence

$\int_0^\infty e^{-t} t^{s-1}\,dt$ is a holo. on $S_{\delta,M}$, $\forall\ 0<\delta<M<\infty$

$\Rightarrow \int_0^\infty e^{-t} t^{s-1}\,dt$ define a holo on $\{Re(s)>0\}$.

Clearly, when restricted to real $s>0$, $\int_0^\infty e^{-t} t^{s-1}\,dt = \Gamma(s)$,

$\therefore \int_0^\infty e^{-t} t^{s-1}\,dt$ is the required extension.

$\divideontimes$