§3 Infinite Products

3.1 Generalities

Def Given $\{a_n\}_{n=1}^{\infty}$ ($a_n \in \mathbb{C}$), we say that the <u>infinite product</u> (a just product)

$$\prod_{n=1}^{\infty} (1+a_n) \quad \underline{\text{converges}}$$

if $\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n)$ exists.

<u>Remark</u>: $\prod_{n=1}^{N} (1+a_n)$ is called the N-term partial product

<u>Prop 3.1</u>: $\sum |a_n| < \infty \Rightarrow \prod_{n=1}^{\infty} (1+a_n)$ converges

In this case, $\prod_{n=1}^{\infty} (1+a_n) = 0 \Leftrightarrow \exists\, n_0$ s.t. $1+a_{n_0} = 0$

<u>Pf</u>: $\sum |a_n| < \infty \Rightarrow |a_n| < \frac{1}{2}$ for sufficiently large n

$\Rightarrow$ for suff. large n,

$$\log(1+a_n) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a_n^k}{k}$$ is well-defined and satisfies

$$1+a_n = e^{\log(1+a_n)}$$ (actually holds $\forall\, |a_n| < 1$)

Hence $\displaystyle\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N}\log(1+a_n)}$.

By the definition of $\log(1+a_n)$, we have for sufficiently large n,

$$|\log(1+a_n)| \leq 2|a_n| \quad \text{for } |a_n|<\frac{1}{2}$$

$$\Rightarrow \quad \sum_{n\gg 1}^{N} |\log(1+a_n)| \leq 2\sum_{n\gg 1}^{N}|a_n|$$

$$\sum|a_n|<\infty \quad \Rightarrow \quad \sum_{n=1}^{N}\log(1+a_n) \text{ converges absolutely}$$

$$\therefore \quad \lim_{N\to+\infty}\prod_{n=1}^{N}(1+a_n) = e^{\sum_{n=1}^{\infty}\log(1+a_n)} \text{ exists.}$$

In this case, if $\exists\, n_0$ s.t. $1+a_{n_0}=0$, then

$\displaystyle\lim_{N\to+\infty}\prod_{n>n_0}^{N}(1+a_n)$ exists and

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{n_0}(1+a_n)\cdot\lim_{N\to\infty}\prod_{n>n_0}^{N}(1+a_n) = 0$$

Since $\sum|a_n|<\infty$, if $1+a_n\neq 0$, $\forall n$.

Then
$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n) = e^{\sum_{n=1}^{\infty}\log(1+a_n)} \neq 0 . \#$$

<u>Prop 3.2</u> Suppose $\{F_n(z)\}$ is a seq. of holo. functions on Ω (open).

If $\exists$ $C_n > 0$ such that

$$\begin{cases} \sum C_n < \infty & \text{and} \\ |F_n(z) - 1| \le C_n, & \forall z \in \Omega, \end{cases}$$

then

(i) $\prod_{n=1}^{\infty} F_n(z)$ converges <u>uniformly</u> in Ω to a holo. function $F(z)$.

(ii) If $F_n(z) \neq 0$, $\forall z \in \Omega$, $\forall n$, then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Pf: Write $F_n(z) = 1 + a_n(z)$.

Then by assumption $|a_n(z)| \le C_n$

and hence $\sum a_n(z)$ uniformly absolute converges on Ω.

By the same argument, as $N \to +\infty$,

$$G_N(z) = \prod_{n=1}^{N} F_n(z) \longrightarrow F(z) = e^{\sum_{n=1}^{\infty} \log(1 + a_n(z))} \quad \text{(uniformly)}$$

which has to be holomorphic on Ω. This proves (i).

For (ii), (Thm 5.3 of Ch 2)

$G_N \to F$ uniformly $\Rightarrow$

$$G_N' \to F' \text{ uniformly on any cpt subset } K \subset \Omega$$

By Prop 3.1, the limit $F(z) \neq 0$, $\forall z \in \Omega$.

Hence $\forall$ cpt. subset $K \subset \Omega$, $\exists \delta > 0$ s.t. $|G_N(z)| \geq \delta$. (for N large)

$$\therefore \quad \sum_{n=1}^{N} \frac{F_n'(z)}{F_n(z)} = \frac{G_N'(z)}{G_N(z)} \to \frac{F'(z)}{F(z)} \text{ uniformly on } K.$$

Since $K \subset \Omega$ is arbitrary, we have $\dfrac{F'(z)}{F(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$.

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3.2 Example: the product formula for the sine function

$$\boxed{\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \text{——— (3)}}$$

We'll prove it by showing that

$$\boxed{\pi \cot \pi z = \lim_{N \to +\infty} \sum_{|n| \leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad \text{——— (4)}}$$

Remarks: (i) Formula (4) holds for $z \in \mathbb{C} \setminus \mathbb{Z}$ only

(ii) $\lim_{N\to+\infty} \sum_{|n|\le N} \frac{1}{z+n}$ is the principal value of $\sum_{n=-\infty}^{\infty} \frac{1}{z+n}$,

other arrangement may not converges.

<u>Pf of (3) by (4).</u>

Write $\quad G(z) = \frac{\sin \pi z}{\pi}$

$$P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$P(z)$ is well-defined since $\left|\frac{-z^2}{n^2}\right| = \frac{|z|^2}{n^2} \le \frac{R^2}{n^2}$, $\forall z \in \{|z|<R\}$

Prop 3.2 $\Rightarrow$

$\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ and hence $P(z)$ is well-defined on $\{|z|<R\}$.

Since $R>0$ is arbitary, $P(z)$ is entire.

Again by Prop 3.2, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \cot \pi z \qquad \text{by formula (4).}$$

Hence for $z \in \mathbb{C} \setminus \mathbb{Z}$

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)}\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right]$$

$$= \frac{P(z)}{G(z)}\left[\pi \cot \pi z - \frac{\cos \pi z}{\left(\frac{\sin \pi z}{\pi}\right)}\right] = 0$$

Since $\mathbb{C}\setminus\mathbb{Z}$ is connected, $P(z) = cG(z)$ for some constant c.

(and clearly extends to whole $\mathbb{C}$)

Letting $z \to 0$ in $\frac{P(z)}{z} = c\,\frac{G(z)}{z}$ (near, but $\neq 0$),

i.e. $\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = c\,\frac{\sin \pi z}{\pi z}$, we have $c = 1$. $\#$