

Thm 2.4 (Poisson Summation Formula)

$$\text{If } f \in \mathcal{F}, \text{ then } \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Pf: $f \in \mathcal{F} \Rightarrow f \in \mathcal{F}_a$ for some $a > 0$.

$\Rightarrow f$ holo. on $S_a = \{x+iy : |y| < a\}$

Consider $g(z) = \frac{f(z)}{e^{2\pi iz} - 1}$ on S_a .

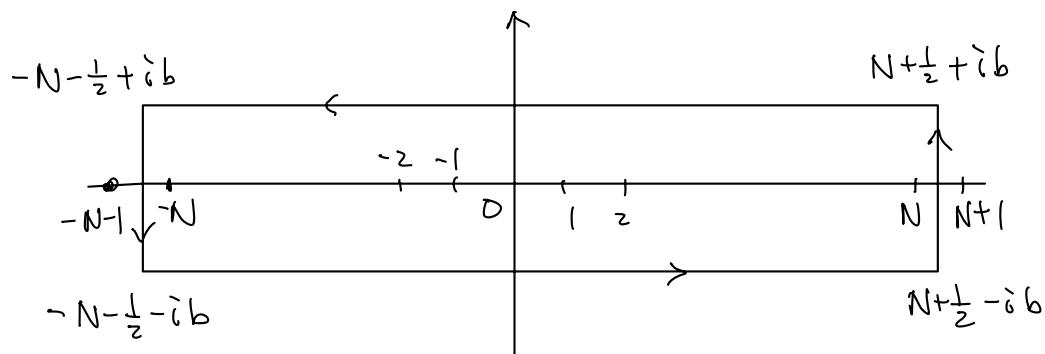
It is easy to see $\frac{1}{e^{2\pi iz} - 1}$ has simple pole at $n \in \mathbb{Z}$

$$\text{with } \operatorname{res}_n \frac{1}{e^{2\pi iz} - 1} = \frac{1}{2\pi i} \quad (\text{Ex!})$$

Hence $g(z) = \frac{f(z)}{e^{2\pi iz} - 1}$ has simple pole at $n \in \mathbb{Z}$

with $\operatorname{res}_n g = \frac{f(n)}{2\pi i}$ $\begin{cases} \text{except } f(n)=0, \text{ where} \\ g \text{ has a removable singularity} \\ \Rightarrow \text{no contribution to the} \\ \text{contour integral.} \end{cases}$

Applying Residue Formula (Cor 2.3 of Ch 3 of Text) to the contour γ_N , $N \in \mathbb{Z}^+$, as in the figure, for $0 < b < a$,



we have

$$2\pi i \sum_{|n| \leq N} \operatorname{res}_n g = \int_{\gamma_N} g(z) dz$$

i.e.

$$\sum_{|n| \leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi iz} - 1} dz.$$

Note that $f \in \mathcal{F}_a \Rightarrow \exists A > 0$ s.t. $|f(z)| \leq \frac{A}{|1 + \operatorname{Re}(z)|^2}$

$$\Rightarrow |f(n)| \leq \frac{A}{|1 + n|^2} \quad \forall n \in \mathbb{Z}$$

$\therefore \sum_{|n| \leq N} f(n) \rightarrow \sum_{n \in \mathbb{Z}} f(n)$ as $N \rightarrow +\infty$.

And $\left| \int_{\pm(N+\frac{1}{2})-ib}^{\pm(N+\frac{1}{2})+ib} \frac{f(z)}{e^{2\pi iz}-1} dz \right| \leq \frac{C}{N^2} \quad (N \in \mathbb{Z}^+) \quad (\text{Ex!})$

for some constant C depending on A and b only

Hence letting $N \rightarrow +\infty$ in $\sum_{|n| \leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi iz}-1} dz,$

we have

$$\sum_{n \in \mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi iz}-1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz}-1} dz$$

where $L_1 = \{x+iy : y = -b\}$ oriented left to right
 $L_2 = \{x+iy : y = b\}$ oriented left to right.

Note that on L_1 , $|e^{2\pi iz}| = |e^{2\pi i(x-ib)}| = e^{2\pi b} > 1$

$$\begin{aligned}\therefore \frac{1}{e^{2\pi i z} - 1} &= \frac{1}{e^{2\pi i z}} \cdot \frac{1}{1 - e^{-2\pi i z}} \\ &= e^{-2\pi i z} \sum_{k=0}^{\infty} e^{2\pi i k z}\end{aligned}$$

Similarly on L_2 , $|e^{2\pi i z}| = e^{-2\pi b} < 1$

$$\frac{1}{e^{2\pi i z} - 1} = - \sum_{k=0}^{\infty} e^{2\pi i k z}$$

$$\begin{aligned}\therefore \sum_{n \in \mathbb{Z}} f(n) &= \int_{L_1} f(z) e^{-2\pi i z} \sum_{k=0}^{\infty} e^{2\pi i k z} dz \\ &\quad + \int_{L_2} f(z) \sum_{k=0}^{\infty} e^{2\pi i k z} dz\end{aligned}$$

Since $|f(z)| \leq \frac{A}{(1+|Rez|)^2}$, both $\int_{L_1} + \int_{L_2}$ can be interchanged

with $\sum_{k=0}^{\infty}$, and we have

$$\begin{aligned}\sum_{n \in \mathbb{Z}} f(n) &= \sum_{k=0}^{\infty} \int_{L_1} f(z) e^{-2\pi i (k+1)z} dz \\ &\quad + \sum_{k=0}^{\infty} \int_{L_2} f(z) e^{2\pi i k z} dz\end{aligned}$$

Then using $(*)_1$ & $(*)_2$ in the proof of Thm 2.1, we have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k=0}^{\infty} \hat{f}(k+1) + \sum_{k=0}^{\infty} \hat{f}(-k) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$$

XX

Applications of Poisson summation formula

(1) For $t > 0$, define the theta function by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

$$\text{Then } \vartheta(t) = t^{-\frac{1}{2}} \vartheta\left(\frac{1}{t}\right), \quad \forall t > 0.$$

Pf: This follows from a more general formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\pi \frac{n^2}{t}} e^{2\pi i na} \quad \text{for } a \in \mathbb{R}.$$

To prove this, we observe that by Eg 1 of Ch 2,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$$

(i.e. Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi \xi^2}$.)

(To be cont'd)