§4 The argument principle and applications

Thm 4.1 & Cor 4.2 (Argument Principle)

Suppose f mero. in an open set containing a simple closed (+ve oriented) piecewise smooth curve γ and $\mathrm{int}(\gamma)$.

If f has neither zeros nor poles on γ,

then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z - P$$

where Z = number of zeros of f in $\mathrm{int}(\gamma)$ &

P = number of poles of f in $\mathrm{int}(\gamma)$

Thm 4.3 (Rouché's Theorem)

Suppose f & g are holo in an open set containing a simple closed piecewise smooth curve γ and $\mathrm{int}(\gamma)$. If

$$|f(z)| > |g(z)| \quad \forall z \in \gamma,$$

then f and $f+g$ have the same number of zeros in $\mathrm{int}(\gamma)$.

Thm 4.4 (Open Mapping Theorem)

If f holo on a region Ω & $f \not\equiv$ const., then f is open.

(i.e. f maps open sets to open sets.)

Thm 4.5 (Maximum modulus principle)

If f holo on a region Ω & $f \not\equiv$ const., then

$|f|$ cannot attain a maximum in Ω.

For simplicity, sometimes we just say "maximum of f" for "maximum of $|f|$"

cpx-valued.

Cor 4.6 Suppose Ω is a region with compact closure $\overline{\Omega}$.

If f holo. on Ω & continuous on $\overline{\Omega}$, then

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \overline{\Omega} \setminus \Omega} |f(z)|$$

§5 Homotopies and Simply Connected Domains

Def: Ω open in $\mathbb{C}$; $\gamma_0(t)$ & $\gamma_1(t)$, $t\in[a,b]$, curves in Ω with common end points, i.e. $\gamma_0(a)=\gamma_1(a)=\alpha$; $\gamma_0(b)=\gamma_1(b)=\beta$.

γ_0 & γ_1 are said to be homotopic in Ω if $\exists$ continuous map $H(s,t):[0,1]\times[a,b]\to\Omega$ such that

$$H(0,t)=\gamma_0(t) \quad \& \quad H(1,t)=\gamma_1(t), \quad \forall t\in[a,b].$$

and $$H(s,a)=\alpha \quad \& \quad H(s,b)=\beta, \quad \forall s\in[0,1]$$

Remark: Usually think of $H(s,t)=\gamma_s(t)$ as a family of curves in Ω with the same end points $\gamma_s(a)=\alpha$ & $\gamma_s(b)=\beta$

s.t. for $s=0$ & 1 they are the original two curves γ_0 & γ_1.

Hence "one curve can be deformed continuously into the other curve without ever leaving Ω".

Thm 5.1 If f holo. in Ω, then

$$\int_{\gamma_0} f(z)\,dz=\int_{\gamma_1} f(z)\,dz$$

provided γ_0 and γ_1 are homotopic in Ω.

Def A region $\Omega \subset \mathbb{C}$ is simply connected if any two curves in Ω with the same end points are homotopic in Ω.

Thm 5.2 & Cor 5.3 If f is holomorphic in a simply connected domain Ω, then

(1) $\exists\ F : \Omega \to \mathbb{C}$ holo s.t. $F' = f$;

(2) $\int_\gamma f(z)\,dz = 0 \quad \forall$ closed curve γ in Ω.

§6 The Complex Logarithm

Thm 6.1 Suppose Ω is simply connected,

$$1 \in \Omega \text{ but } 0 \notin \Omega.$$

Then $\exists$ a branch of the logarithm $F(z) = \log_\Omega z$ s.t.

(i) F is holo. in Ω

(ii) $e^{F(z)} = z, \ \forall z \in \Omega$

(iii) $F(r) = \log r \quad \forall\ r \in \mathbb{R}$ and near to 1.

- Principal branch of the logarithm

$$\begin{cases} \Omega = \mathbb{C} \setminus (-\infty, 0], \\ \log z = \log r + i\theta \ \text{ with } \boxed{|\theta| < \pi} \ \text{ and } z = re^{i\theta}. \end{cases}$$

Thm 6.2 • Ω = simply connected region,

• $f: \Omega \to \mathbb{C}$ holo. & $f(z) \neq 0, \forall z \in \Omega$.

Then $\exists\, g: \Omega \to \mathbb{C}$ holo. s.t.

$$e^{g(z)} = f(z) \qquad \forall z \in \Omega.$$

(g is denoted by $\log f$)

§7 Fourier Series and Harmonic Functions

Thm 7.1 $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges in $D_R(z_0)$.

Then $\forall\, r \in (0, R)$,

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{-in\theta}\, d\theta = \begin{cases} a_n r^n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Remarks (i) This is just the Cauchy integral formula applies to the circle

$$\gamma(\theta) = z_0 + re^{i\theta}, \quad \theta \in [0, 2\pi]$$

(ii) The LHS is the Fourier coefficients (up to a const.)

of the 2π-periodic function $f(z_0 + re^{i\theta})$ (for fixed r.)

<u>Cor 7.2 & 7.3</u> $f = u + iv$ holo. in $D_R(z_0)$,

Then $$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \quad \forall\ 0 < r < R.$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta, \quad \forall\ 0 < r < R$$

These are <u>mean-value property</u> for <u>holomorphic</u> and <u>harmonic</u> function respectively.

(End of Review)

Ch4 The Fourier Transform

§1 The Class $\mathcal{F}$

Def: $\forall a>0$, let $S_a = \{z\in\mathbb{C} : |\mathrm{Im}(z)|<a\}$ (a horizontal strip)

Then

$$\mathcal{F}_a = \left\{ f : S_a \to \mathbb{C} : \begin{array}{l} f \text{ holo. on } S_a \text{ and } \exists A>0 \text{ s.t.} \\ |f(x+iy)| \le \dfrac{A}{1+x^2},\ \forall x\in\mathbb{R}\ \&\ |y|<a \end{array} \right\}$$

and $\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$

Remark: For a fixed y, with $|y|<a$, the condition that

$$\exists A>0 \text{ s.t. } |f(x+iy)| \le \frac{A}{1+x^2},\quad \forall x\in\mathbb{R}$$

is usually referred as "moderate decay" on the horizontal line $\mathrm{Im}(z)=y$.

Hence, $f\in\mathcal{F}_a$ are moderate decay for each horizontal line $\mathrm{Im}(z)=y$, uniformly in $|y|<a$.

egs (i) Clearly $f(z)=e^{-\pi z^2}\in\mathcal{F}_a$, $\forall a>0$ (Ex!)

(ii) $\forall\, c>0$, the function

$$f(z) = \frac{1}{\pi}\,\frac{c}{c^2+z^2} \in \mathcal{F}_a, \ \forall\, a \in (0,c). \ \text{(Ex!)}$$

Clearly, $f(z) \notin \mathcal{F}_a$ for $a \geq c$ as $z = \pm ci$ are poles.

Remarks: (1) For integer $n \geq 1$, $f \in \mathcal{F}_a \Rightarrow f^{(n)} \in \mathcal{F}_b$, $\forall\, 0<b<a$.

(Ex 2 of Ch 4 of Text)

(2) Many results in this chapter remain unchanged under the following weaker condition: $(\varepsilon > 0)$

$$\exists A>0 \text{ s.t. } |f(x+iy)| \leq \frac{A}{1+|x|^{1+\varepsilon}} \quad \forall\, x \in \mathbb{R} \ \&\ |y|<a.$$

(Omitted)

§2 Action of the Fourier Transform on $\mathcal{F}$

Def Let $f: \mathbb{R} \to \mathbb{C}$. The Fourier transform of f is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx, \quad \xi \in \mathbb{R}$$

For $f \in \mathcal{F}_a$, we consider the Fourier transform of $f(x)$, i.e. when $y=0$. Then we have

$\boxed{\text{Thm 2.1 If } f \in \mathcal{F}_a \text{, for some } a>0 \text{, then } \exists B>0 \text{ s.t. } |\hat{f}(\xi)| \leq B e^{-2\pi b|\xi|},\ \forall\ 0 \leq b < a.}$

Pf: $|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)|\,dx$ since $x, \xi \in \mathbb{R}$

$\leq \int_{-\infty}^{\infty} \frac{A}{1+x^2}\,dx = B$ which is bounded,

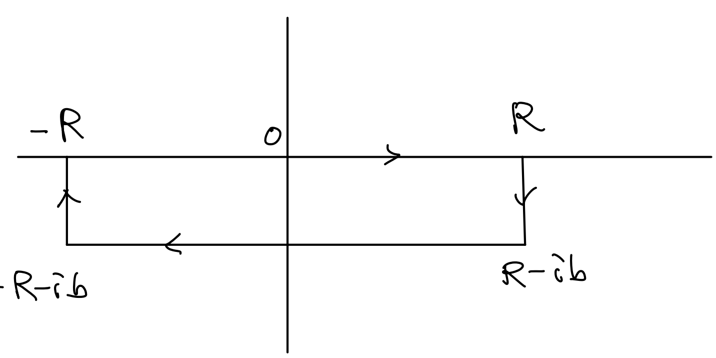
$\Rightarrow \quad |\hat{f}(\xi)| \leq B e^{-2\pi b|\xi|}$ holds for $b=0$.

For $0<b<a$:

If $\xi > 0$, consider the contour integral of the holo. function

$g(z) = f(z) e^{-2\pi i z \xi}$ in S_a along the contour

which is the boundary of the rectangle $[-R,R] \times [-b,0]$ $(R>0)$



On the vertical edge $[-R-ib, -R]$,

using parametrization $z(t) = -R - it$, $t \in [0, b]$ (reverse direction)

$$\left| \int_{-R-ib}^{-R} g(z)\,dz \right| \le \int_0^b \left| f(-R-it)\, e^{-2\pi i(-R-it)\xi} \right| dt$$

$$\le \int_0^b \frac{A}{1+R^2}\, e^{-2\pi \xi t}\, dt \qquad \text{since } \xi > 0$$

$$= \frac{A}{1+R^2} \int_0^b e^{-2\pi\xi t}\, dt \longrightarrow 0 \quad \text{as } R \to +\infty.$$

Similarly $\left| \int_R^{R-ib} g(z)\,dz \right| \le \frac{A}{1+R^2} \int_0^b e^{-2\pi\xi t}\, dt \longrightarrow 0$ as $R \to +\infty$.

Therefore, Cauchy theorem $\Rightarrow$

$$\left| \int_{-R}^{R} f(x) e^{-2\pi i x\xi}\, dx - \int_{-R}^{R} f(x-ib) e^{-2\pi i (x-ib)\xi}\, dx \right| \le \frac{2A}{1+R^2} \int_0^b e^{-2\pi\xi t}\, dt.$$

Letting $R \to +\infty$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\, dx$$

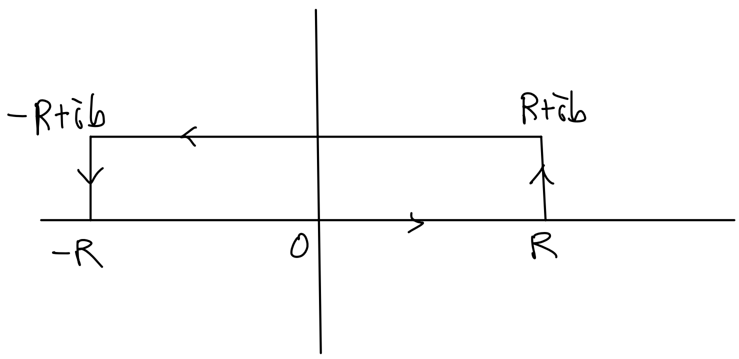
$$= \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i (x-ib)\xi}\, dx \quad \text{——— } (*),$$

$$\Rightarrow \quad |\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x-ib)|\, e^{-2\pi b\xi}\, dx \qquad (\xi > 0)$$

$$\leq \left(\int_{-\infty}^{\infty} \frac{A}{1+x^2} dx \right) e^{-2\pi b \xi}$$

$$= B e^{-2\pi b |\xi|} \qquad (\xi > 0)$$

For $\xi < 0$, consider similarly the contour integral of $g(z)$ along:



$$\Rightarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib) e^{-2\pi i (x+ib)\xi} dx \qquad (*)_2 \quad (\text{Ex!})$$

and hence the result. $\#$

<u>Remark</u>: Therefore, if $f \in \mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$, then

$|\hat{f}(\xi)|$ decay exponentially as $|\xi| \to +\infty$,

in particular, it is <u>rapid decay at infinity</u> (i.e. decay faster than any $|\xi|^{-N}$, $\forall N > 0$. More precisely $o(\frac{1}{|\xi|^N})$, $\forall N > 0$.)

Thm 2.2 (Fourier Inversion Formula)

If $f \in \mathcal{F}$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad \forall x \in \mathbb{R}$$

The proof needs a lemma:

Lemma 2.3 If $A > 0$ & $B \in \mathbb{R}$, then

$$\int_0^{\infty} e^{-(A+iB)\xi} d\xi = \frac{1}{A+iB}$$

Pf:

Note $A>0, B \in \mathbb{R} \Rightarrow |e^{-(A+iB)\xi}| = e^{-A\xi}$, for $\xi \in (0,\infty)$.

Hence the improper integral converges.

$$\therefore \int_0^{\infty} e^{-(A+iB)\xi} d\xi = \lim_{R \to +\infty} \int_0^{R} e^{-(A+iB)\xi} d\xi$$

$$= \lim_{R \to +\infty} \left[\frac{e^{-(A+iB)\xi}}{-(A+iB)} \right]_0^R = \frac{1}{A+iB}$$

#

Pf of Thm 2.2 (Fourier Inversion Formula)

Note that $f \in \mathfrak{F} \Rightarrow f \in \mathfrak{F}_a$ for some $a>0$.

Then by equations $(*)_1$ & $(*)_2$ in the proof of Thm 2.1,

(eqt (1) in the text)

If $\xi>0$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)\, e^{-2\pi i (x-ib)\xi}\, dx$, $\forall\ 0<b<a$.

If $\xi<0$, $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib)\, e^{-2\pi i (x+ib)\xi}\, dx$, $\forall\ 0<b<a$

As sign of ξ is important in the proof, we write

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x\xi} d\xi + \int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi$$

and work on the integrals individually:

$$\int_0^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i (u-ib)\xi} du \right) e^{2\pi i x\xi} d\xi$$

Since $|f(u-ib)| \leq \frac{A}{1+u^2}$, (for some $A>0$)

the (iterated) integrals are absolute convergence.

Hence Fubini $\Rightarrow$

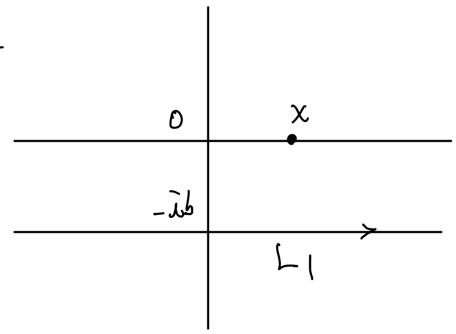
$$\int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = \int_{-\infty}^{\infty} f(u-ib) \int_0^\infty e^{-2\pi i (u-ib)\xi} e^{2\pi i x\xi}\, d\xi\, du$$

$$= \int_{-\infty}^{\infty} f(u-ib) \left(\int_0^\infty e^{-2\pi i (u-x-ib)\xi}\, d\xi \right) du$$

(Lemma 2.3) $$= \int_{-\infty}^{\infty} f(u-ib) \frac{1}{2\pi b + 2\pi i (u-x)}\, du$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{(u-ib)-x}\, du$$

$$= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x}\, d\zeta$$



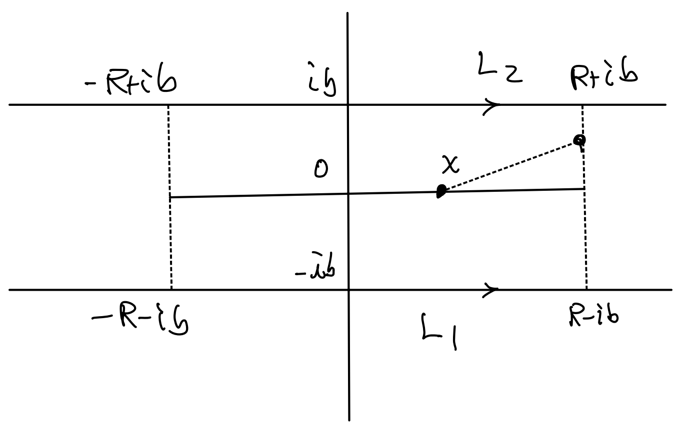
The contour integral of $\frac{f(z)}{z-x}$ (x fixed)

along the horizontal line $y=-b$ (from left to right)

Similarly for $\xi < 0$,

$$\int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

$$= -\frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x}\, d\zeta$$



where $L_2 : \{y = b\}$ from left to right. (Check!)

Note that

$$\left| \int_{R-ib}^{R+ib} \frac{f(\zeta)}{\zeta - x} \, d\zeta \right| \leq 2b \, \frac{A}{1+R^2} \cdot \frac{1}{R-x} \qquad \text{for } R > x.$$

$$\longrightarrow 0 \quad \text{as} \quad R \to +\infty$$

Similarly,

$$\left| \int_{-R-ib}^{-R+ib} \frac{f(\zeta)}{\zeta - x} \, d\zeta \right| \to 0 \quad \text{as } R \to +\infty.$$

Cauchy integral formula, by letting $R \to +\infty$,

$$f(x) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} \, d\zeta - \frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} \, d\zeta$$

$$= \int_0^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi + \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) \, e^{2\pi i x \xi} \, d\xi.$$

#