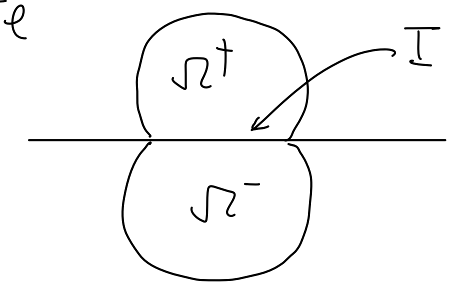
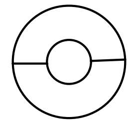
5.4 Schwarz reflection principle

Def: • An open set Ω (region) in $\mathbb{C}$ is symmetric with respect to the real line if

$$z \in \Omega \Leftrightarrow \bar{z} \in \Omega .$$

If Ω is symmetric wrt $\mathbb{R}$-line, we denote



- $\Omega^+ = \{ z = x+iy \in \Omega : y > 0 \}$
- $\Omega^- = \{ z = x+iy \in \Omega : y < 0 \}$.
- $I = \Omega \cap \mathbb{R}$. (I may not be a single interval )

Then $\Omega = \Omega^+ \cup I \cup \Omega^-$

Thm 5.5 (Symmetry principle)

If $f^+ : \Omega^+ \to \mathbb{C}$, $f^- : \Omega^- \to \mathbb{C}$ holo. such that $f^\pm$ extend continuously to $\Omega^\pm \cup I$ with

$$f^+(x) = f^-(x), \ \forall x \in I,$$

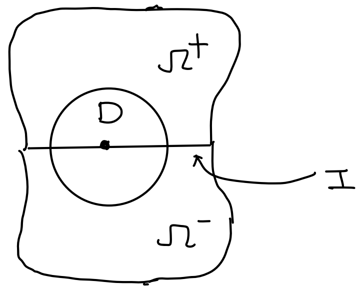
then

$$f(z) = \begin{cases} f^+(z) & , \ z \in \Omega^+ \\ f^+(z) = f^-(z) & , \ z \in I \\ f^-(z) & , \ z \in \Omega^- \end{cases}$$

is holo. on Ω.

Pf: Clearly only need to show that

f is holo at points of I.



Hence, we only need to consider

a disc $D \subset \bar{D} \subset \Omega$ s.t.

its center $\in I$.

Then D is symmetric wrt $\mathbb{R}$-line too

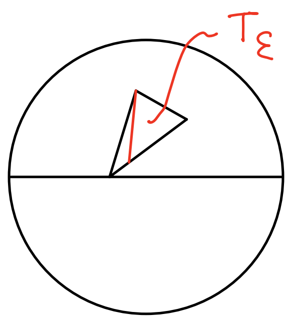
Consider triangle $T \subset D$.

If $T \subset D^+$ or D^-,

then Cauchy's Thm $\Rightarrow$ $\int_{\partial T} f\,dz = 0$.

If $T \cap I \neq \emptyset$, then

Case 1 $T \cap I =$ a vertex of T

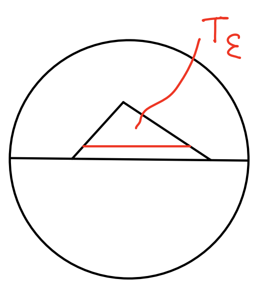


Approximate by a $T_\varepsilon \subset D^+$ or D^-

Then uniform continuity of f & $\int_{\partial T_\varepsilon} f\,dz = 0$, $\forall \varepsilon > 0$

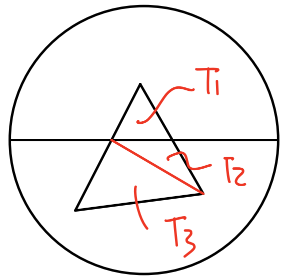
$$\Rightarrow \int_{\partial T} f\,dz = 0$$

Case 2 $T \cap I =$ an edge of T



Same argument as in Case 1.

Case 3 $T \cap D^+ \neq \emptyset$ and $T \cap D^- \neq \emptyset$



Then $T \cap I$ divides T into triangle or polygon completely contained in $D^+ \cup I$ or $D^- \cup I$.

If it is a triangle, apply Case 2.

If it is a polygon, subdivide the polygon into triangles as in Cases 1 & 2. Then using results in cases 1 & 2 and by the cancellation of the integrals along the common edges, we have

$$\int_{\partial T} f\,dz = 0.$$

By Morera's Thm (Thm 5.1), f is holo. on I. ⋕

Thm 5.6 (Schwarz Reflection Principle)

Let • Ω (region) be symmetric wrt $\mathbb{R}$-line.

• $f : \Omega^+ \to \mathbb{C}$ is holomophic and extends continuously to I such that

• $f(x) \in \mathbb{R}$, $\forall x \in I$.

Then $\exists$ $F : \Omega \to \mathbb{C}$ holomorphic such that

$$F|_{\Omega^+} = f.$$

(In fact, F is unique by Thm 4.8 (assuming connectedness of Ω))

Pf: Define $f^-(z) = \overline{f(\bar{z})}$ for $z \in \Omega^-$.

Then it is easy to check

• $f^- : \Omega^- \to \mathbb{C}$ is holomophic

• f^- extends continuously to I

and $\forall x \in I$, $f^-(x) = \overline{f(\bar{x})} = \overline{f(x)} = f(x)$ as $f(x) \in \mathbb{R}$

By Thm 5.5 (Symmetric principle)

$$F(z) = \begin{cases} f(z), & z \in \Omega^+ \cup I \\ f^-(z) = \overline{f(\bar{z})}, & z \in \Omega^- \end{cases}$$ is holomophic on Ω.

and clearly $F|_{\Omega^+} = f$. ※

§5.5 Runge's Approximation Theorem

Omitted.

Ch 3 Meromophic Functions and the Logarithm

§1 Zeros and Poles

<u>Thm 1.1 & Thm 1.2</u>
- Ω open in $\mathbb{C}$, $z_0 \in \Omega$,
- f holo. in Ω or $\Omega \setminus \{z_0\}$

Then in a nbd. of z_0, $\exists$ holo function g and integer $n \geq 1$ s.t.

$$f(z) = \begin{cases} (z-z_0)^n g(z) & \Leftrightarrow \quad z_0 \text{ is a zero } \& \ f \text{ holo. in } \Omega \\ (z-z_0)^{-n} g(z) & \Leftrightarrow \quad z_0 \text{ is a pole } \& \ f \text{ holo in } \Omega \setminus \{z_0\} \end{cases}$$

- multiplicity of zeros and poles
- simple zero and simple poles
- Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, <u>isolated singularities</u>
- Principal part at a pole
- Residue at a pole

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \overset{\text{residue}}{\frac{a_{-1}}{z-z_0}}}_{\text{principal part}} + \underset{\text{holo in a nbd of } z_0}{G(z)}$$

§2 The Residue Formula

Thm 2.1, Cor 2.2 & Cor 2.3 (Residue formula)

Suppose f holo in an open set containing

a simple closed (+ve oriented) piecewise smooth curve γ

and $\mathrm{int}(\gamma)$, except for poles at $z_1, \dots, z_N \in \mathrm{int}(\gamma)$.

Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f$$

§3 Singularities and meromorphic functions

Thm 3.1 (Riemann's Theorem on removable singularities)

Suppose Ω open in $\mathbb{C}$, $z_0 \in \Omega$.

$f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic.

If f is bounded on $\Omega \setminus \{z_0\}$,

then z_0 is a removable singularity

(i.e. f can be extended to a holomorphic function on Ω)

$\Rightarrow$ For isolated singularities either
- removable (f bdd near z_0)
- pole ($|f| \to +\infty$ as $z \to z_0$)

or
- essential singularities

Thm 3.3 (**Casorati–Weierstrass**)

If $f: D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ holo. and has an **essential singularity** at z_0, then

$$f(D_r(z_0) \setminus \{z_0\}) \text{ dense in } \mathbb{C}.$$

$$\left.\begin{array}{l}\bullet \text{ extended complex plane,} \\ \bullet \text{ rational functions} \\ \bullet \text{ Riemann sphere} \\ \bullet \text{ Stereographic projection}\end{array}\right\} \text{self-reading.}$$