HW 6A. $m(E) < +\infty$, $E = \emptyset$, $0 \cdot \infty \overset{\text{convention}}{=\!=\!=} 0$.

1. Let $\varphi \in \mathcal{S}_0 \setminus \{0\}$ and list non-zero distinct values of φ: $b_1, b_2, \cdots b_N$ (not repeat) and $b_0 := 0$. Let $B_j := \varphi^{-1}(b_j)$ $\forall j$.

Define
$$\int_{-\infty}^{\infty} \varphi = \sum_{j=0}^{N} b_j m(B_j) \text{ and } \int_E \varphi := \int_{-\infty}^{\infty} (\varphi \chi_E).$$

Show that if $\alpha_i \in \mathbb{R} \setminus \{0\}$, $\emptyset \neq E_i \in \mathfrak{M}$
$\varphi := \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ and, additionally $\{E_i : i = 1, 2 \cdots n\}$ disjoint
then $\{\alpha_1, \alpha_2, \cdots \alpha_n\} = \{b_1, b_2, \cdots, b_N\}$ and
$$B_j = \bigcup_{\alpha_i = b_j} E_i \quad \forall j.$$

Show further that $b_j m(B_j) = \sum_{\alpha_i = b_j} \alpha_i \cdot m(E_i)$ and
$$\int_{-\infty}^{\infty} \varphi = \sum_{i=1}^{n} \alpha_i m(E_i).$$

Define
$$\int_{-\infty}^{\infty} \underset{\uparrow \text{ zero-function}}{0} = 0$$

If no confuse one writes $\int \varphi$ for $\int_{-\infty}^{\infty} \varphi$.

2*. Let $\psi \in \mathcal{S}_0 \setminus \{0\}$, "represent ψ canonically":

$$\psi = \sum_{i=1}^{M} c_i \chi_{\psi^{-1}(c_i)} \qquad (c_0 := 0)$$

with distinct non-zero values $c_1, c_2, \cdots, c_M$.

Show that

$$\varphi + \psi = \sum_{i,j} (b_j + c_i) \chi_{\varphi^{-1}(b_j) \cap \psi^{-1}(c_i)}$$

and use Q1 to show that $\int_{-\infty}^{\infty} (\varphi + \psi) = \int_{-\infty}^{\infty} \varphi + \int_{-\infty}^{\infty} \psi$

and that if $\Delta = \Delta_1 \cup_0 \Delta_2$ with $\Delta_1, \Delta_2 \in \mathfrak{M}$ then $\int_\Delta \varphi = \int_{\Delta_1} \varphi + \int_{\Delta_2} \varphi$ (Hint: $\varphi \chi_\Delta = \varphi \chi_{\Delta_1} + \varphi \chi_{\Delta_2}$).

Show further that

$\varphi \mapsto \int_{-\infty}^{\infty} \varphi$ is linear on $\mathcal{S}_0$

and is $\uparrow$: $\underset{\varphi, \psi \in \mathcal{S}_0}{\varphi \leq \psi \text{ on } \mathbb{R}} \Rightarrow \int_{-\infty}^{\infty} \varphi \leq \int_{-\infty}^{\infty} \psi$.

Show moreover that if $\varphi \sim \psi$ then $\int_{-\infty}^{\infty} \varphi = \int_{-\infty}^{\infty} \psi$.

3. Show that if $\varphi \leq \psi$ a.e. on $\mathbb{R}$ $(\varphi, \psi \in \mathcal{S}_0)$ and

$$\int_{-\infty}^{\infty} \varphi = \int_{-\infty}^{\infty} \psi$$

then $\varphi \sim \psi$ (Hint: use the linearity and do for the special case $\varphi = 0$ a.e on $\mathbb{R}$)