$f \in \mathcal{MF}^+(E)$: f is nonnegative a.e. on a measurable set E with $m(E) \leq +\infty$. Define

$$\int_E f := \sup\left\{ \int_E h : h \leq f \text{ a.e. on } E,\ h \in \mathcal{BMF}_0(E) \right\}$$

$$= \sup\left\{ \int_E k : 0 \leq k \leq f \text{ a.e. on } E,\ k \in \mathcal{BMF}_0(E) \right\}$$

(so $\int_E f \in [0, \infty]$).

(the last equality holds as $\int_E h \leq \int_E k$ with $k := h \vee 0$)

Short $\int f$ if E is understood.

1. Show that $f \mapsto \int f$ is $\uparrow$ on $\mathcal{MF}^+(E)$ and

$$\int (\alpha f) = \alpha \int f \quad \forall f \in \mathcal{MF}^+(E) \ \& \ \forall \alpha \in [0, \infty).$$

and unchange if f is replaced by an equivalent function.

2. For real nos. show that $-(a \vee b) = (-a) \wedge (-b)$, $a + (b \vee c) = (a+b) \vee (a+c)$, $a - (a \wedge b) = 0 \vee (a-b)$.

State and prove the corresponding results for real-valued functions f, g, h on D with lattice operations $f \vee g$, $f \wedge g$ and linear operations (all pointwisely defined)

3. Let $f, f_1, f_2 \in \mathcal{MF}^+(E)$ with $f = f_1 + f_2$ and $k \in \mathcal{BMF}_0(E)$ be such that $0 \leq k \leq f$ a.e. on E.

Let $k_1 := k \wedge f_1$ and

$$k_2 := k - k_1 \; \left(= 0 \vee (k - f_1) \right)$$

Show that $0 \leq k_i \leq f_i$ and $k = k_1 + k_2$ a.e. on E. Hence show that $H = H_1 + H_2$ where

$$H := \{ h \in \mathcal{BMF}_0(E) : 0 \leq h \leq f_1 + f_2 \text{ a.e. on } E \}$$

and

$$H_i := \{ h_i \in \mathcal{BMF}_0(E) : 0 \leq h_i \leq f_i \text{ a.e. on } E \} \quad (i = 1, 2).$$

Show finally that

$$\int_E (f_1 + f_2) = \int_E f_1 + \int_E f_2 \quad \forall f_1, f_2 \in \mathcal{MF}^+(E).$$

and

$$\int_{\Delta_1 \dot\cup \Delta_2} f = \int_{\Delta_1} f + \int_{\Delta_2} f \quad \forall f \in \mathcal{MF}^+(E)$$

$\forall \Delta_1, \Delta_2$ disjoint measurable subsets of E.

4. Let $f \in \mathcal{MF}^+(E)$ & $A \subseteq B \subseteq E$, with $A, B \in \mathfrak{M}$. Show that $\int_A f \leq \int_B f$.

Let

$$\Delta_{1/n} := \{x \in E : \tfrac{1}{n} \leq f(x)\}$$

and

$$\Delta := \{x \in E : f(x) \neq 0\}$$

Show that if $f \in \mathcal{MF}^+(E)$ and $\int_E f = 0$ then $m(\Delta) = 0$ (Hint:

$$\int_{\Delta_{1/n}} \left(\frac{1}{n}\right) \leq \int_{\Delta_{1/n}} f \leq \int_E f = 0$$

and $\Delta = \bigcup_{n=1}^{\infty} \Delta_{1/n}$. Similarly if $f \in \mathcal{MF}^+(E)$ & $\int_E f < +\infty$ then $m(E_\infty) = 0$ where

$$E_\infty := \{x \in E : f(x) = +\infty\}$$ (Hint: $\forall n$

$$n \cdot m(E_\infty) = \int_{E_\infty} n \leq \int_{E_\infty} f \leq \int_E f \in \mathbb{R})$$

5. Let $f \in \mathcal{L}(E)$: f is a measurable function on E with $m(E) \leq +\infty$ such that $f^+ := f \vee 0$, $f^- = (-f) \vee 0$ are in $\mathcal{MF}^+(E)$ such that $\int_E f^+, \int_E f^- < +\infty$.

(so f^+, f^- are finite-valued a.e. & $f = f^+ - f^-$ a.e.). Define

$$\int_E f = \int_E f^+ - \int_E f^-$$

and show that if $f = g - h$ a.e. on E

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; f^+ - f^-$

and $f, h \in \mathcal{MF}^+(E)$ with

$$\int_E g, \int_E h \in \mathbb{R},$$

then

$$\int_E f = \int_E g - \int_E h$$

(Hint: $\int_E (g + f^-) = \int_E (f^+ + h)$ by Q3)

6^* $f \mapsto \int_E f$ is linear on $\mathcal{L}(E)$ & $\uparrow$

and $\int_E f$ unchange if f is replaced

by an equivalent function