Solutions 3

. . . .

(3.1) Given a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, for each $\theta \in \mathbb{R}$, consider the curve $\gamma_\theta : \mathbb{R} \to \text{Graph}(f)$, defined by

$$\gamma_\theta(t) = (t\cos\theta, t\sin\theta, f(t\cos\theta, t\sin\theta)), \quad \forall t \in \mathbb{R}.$$

Recall from lectures that the first fundamental form with respect to the coordinates $X : \mathbb{R}^2 \to \text{Graph}(f)$, $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$, is given by

$$g = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}.$$

a) For each $\theta \in \mathbb{R}$, show that the length of the curve restricted to the interval $[-1, 1]$ is given by

$$L(\gamma_\theta|_{[-1,1]}) = \int_{-1}^{1} \sqrt{1 + (f_1 \cos\theta + f_2 \sin\theta)^2}\,dt.$$

b) In the case $f(x, y) = x^2 - y^2$, find the values of θ which minimise the length $L(\gamma_\theta|_{[-1,1]})$.

c) In the case $f(x, y) = e^{xy}$, find the values of θ which minimise the length $L(\gamma_\theta|_{[-1,1]})$.

Solution (3.1)

a) Plugging everything into the equation for length we have

$$\begin{aligned} L(\gamma_\theta|_{[-1,1]}) &= \int_{-1}^{1} \sqrt{(1 + f_1^2)\cos^2\theta + (1 + f_2)^2 \sin^2\theta + 2 f_1 f_2 \sin\theta\cos\theta dt} \\ &= \int_{-1}^{1} \sqrt{1 + (f_1\cos\theta + f_2\sin\theta)^2}\,dt. \end{aligned}$$

b) Since $f_1 = 2t\cos\theta$, $f_2 = 2t\sin\theta$, the formula from a) becomes

$$L(\gamma_\theta|_{[-1,1]}) = \int_{-1}^{1} \sqrt{1 + 4t^2\cos^2(2\theta)}\,dt,$$

and so the length is minimal when $\cos 2\theta = 0$, which happens iff $\theta = \pi/4 + k\pi/2$ for some $k \in \mathbb{Z}$.

c) Since $f_1 = ft\cos\theta$, $f_2 = ft\sin\theta$, the formula from a) becomes

$$L(\gamma_\theta|_{[-1,1]}) = \int_{-1}^{1} \sqrt{1 + f^2 t^2 \sin^2(2\theta)}\,dt.$$

Since f is always positive, the length is minimised when $\sin 2\theta = 0$, which happens iff $\theta = k\pi/2$ for some $k \in \mathbb{Z}$.

(3.2) Let $X : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\}$ denote the stereographic coordinates

$$X(u,v) := \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

For each $r > 0$, let

$$\Omega_r := X\left(\{(u,v) \in \mathbb{R}^2 : \sqrt{u^2+v^2} \leq r\} \right) \subseteq \mathbb{S}^2.$$

That is, Ω_r is the image under stereographic coordinates of the the closed disk of radius r centred at the origin in the plane.

a) Calculate the area of the region Ω_r.

b) Find a sequence of numbers $r_n \uparrow \infty$ such that the ratio of the area of Ω_{r_n} to the area of its complement $\overline{\mathbb{S}^2 \setminus \Omega_{r_n}}$ is exactly $n : 1$.

Solution (3.2)

a) It was shown in tutorials that the first fundamental form in these coordinates is

$$g = \begin{pmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{pmatrix},$$

and so

$$dA = \sqrt{\det g|_{(u,v)}}dudv = \frac{4}{(1+u^2+v^2)^2}dudv.$$

Switching to polar coordinates (ρ, θ), we find that

$$\begin{aligned} \int_{\Omega_r} dA &= \int_{u^2+v^2 \leq r^2} \frac{4}{(1+u^2+v^2)^2}dudv \\ &= \int_0^{2\pi} \int_0^r \frac{4\rho}{(1+\rho^2)^2}d\rho d\theta \\ &= 2\pi \left(\frac{-2}{(1+\rho^2)} \bigg|_0^r \right) = 4\pi \left(1 - \frac{1}{1+r^2} \right). \end{aligned}$$

b) Either by taking $r \uparrow \infty$ or otherwise, we have know that the area of $\mathbb{S}^2$ is 4π. Thus, r_n must satisfy

$$4\pi \left(1 - \frac{1}{1+r_n^2} \right) = \int_{\Omega_{r_n}} dA = 4\pi \left(\frac{n}{n+1} \right) = 4\pi \left(1 - \frac{1}{1+n} \right),$$

and so $r_n = \sqrt{n}$.

(3.3) Prove that any surface of revolution is an orientable surface.

Solution (3.3) Suppose S is a surface of revolution given rotating $\gamma : \mathbb{R} \to \mathbb{R}^2$, a closed curve of length L, about the z-axis. If $\gamma(s) = (f(s), y(s))$, then, as in the lecture notes, we can cover S by three charts $X, Y, Z : (0, L) \times (0, 2\pi) \to S$,

$$\begin{aligned} X(s,\theta) &= (f(s)\cos\theta, y(s), f(s)\sin\theta)\,, \\ Y(s,\theta) &= X(s + L/3, \theta + \pi/2) \\ Z(s,\theta) &= X(s + 2L/3, \theta + \pi). \end{aligned}$$

On overlaps, the change of coordinate function is always given by

$$h(s,\theta) = (s + c_1\theta + c_2),$$

for a pair of constants $c_1, c_2 \in \mathbb{R}$, which has Jacobian matrix I_2. Therefore, these charts define the same orientation on every tangent space of S, and S is orientable.

(3.4) Let S be a regular surface. Given a collection of charts $\{X^i : U_i \to S\}_{i\in I}$ over some index I, we say that they form an **atlas** on S if they cover S

$$\bigcup_{i\in I} X^i(U_i) = S.$$

We say an atlas $\{X^i : U_i \to S\}_{i\in I}$ is **orientable** if the charts define a unique orientation on every tangent space of S. Therefore, S is orientable if and only if S admits an orientable atlas.

We consider the collection of all orientable atlases on our surface S

$$\mathcal{A} = \{\mathfrak{a} = \{X^i : U_i \to S\}_{i\in I} : \mathfrak{a} \text{ is an orientable atlas on } S\}.$$

Given two orientable atlases $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{A}$, their union $\mathfrak{a}_1 \cup \mathfrak{a}_2$ is also an atlas on S. We define the relation $\mathfrak{a}_1 \sim \mathfrak{a}_2$ if $\mathfrak{a}_1 \cup \mathfrak{a}_2 \in \mathcal{A}$, i.e. their union is also an orientable atlas.

a) Show that $\sim$ given above is a well-defined equivalence relation on $\mathcal{A}$. In particular, we can then define the space of orientations on S to be the quotient space

$$Or(S) = \mathcal{A}/\sim.$$

We now consider the space of smooth unit normal vector fields

$$\mathcal{N} := \{N : S \to \mathbb{R}^3 : N \text{ is a smooth with } \|N_p\| = 1,\ N_p \perp T_pS,\ \forall p \in S\}.$$

In the lectures (Lemma 3.17) we showed that $Or(S) \neq \emptyset \iff \mathcal{N} \neq \emptyset$.

b) Find a bijection between the sets $Or(S)$ and $\mathcal{N}$.

c) Consider the hyperboloid

$$H := \{(x, y, z) \in \mathbb{R}^3 : -x^2 - y^2 + z^2 = 1\}.$$

It was shown in lectures that H is a regular surface.

How many distinct orientations does H have? That is, what is the cardinality of the set $Or(H)$? Justify your answer.

Solution (3.4)

a) Since $\mathfrak{a} = \mathfrak{a} \cup \mathfrak{a} \in \mathcal{A}$, $\mathfrak{a} \sim \mathfrak{a}$. Since $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{b} \cup \mathfrak{a}$, $\mathfrak{a} \sim \mathfrak{b}$ iff $\mathfrak{b} \sim \mathfrak{a}$. Finally, suppose $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{A}$ with $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{b} \sim \mathfrak{c}$. Fix $p \in S$. Then there exists a chart $X : U \to S$ in $\mathfrak{b}$ about p. By the assumed relations, for any charts in both $\mathfrak{a}$ and $\mathfrak{c}$, the orientations they define on T_pS must be the one defined by X, and hence all charts in $\mathfrak{a} \cup \mathfrak{c}$ define the same orientation on T_pS. As p was arbitrary, $\mathfrak{a} \cup \mathfrak{c} \in \mathcal{A}$, and hence $\mathfrak{a} \sim \mathfrak{c}$.

b) Given an oriented atlas $\mathfrak{a} \in \mathcal{A}$, Lemma 3.17 from the lecture notes produces a well-defined $N \in \mathcal{N}$. Consider this proceedure as a function $F : \mathcal{A} \to \mathcal{N}$.

Note that, in Lemma 3.17, we showed that such a map is surjective: any element of $\mathcal{N}$ comes from an oriented atlas.

Finally, we see that for two elements $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, $F(\mathfrak{a}) = F(\mathfrak{b})$ if and only if, for any chart $X : U \to S$ in $\mathfrak{a}$ and any chart $Y : V \to S$ in $\mathfrak{b}$, $N^X = N^Y$ on the overlap $X(U) \cap Y(V)$, which happens if and only if $\mathfrak{a} \cup \mathfrak{b} \in \mathcal{A}$, or $\mathfrak{a} \sim \mathfrak{b}$.

Therefore, F descends to a well-defined bijection $\tilde{F} : Or(S) \to \mathcal{N}$,

$$\tilde{F}([\mathfrak{a}]) := F(\mathfrak{a}), \quad \forall \mathfrak{a} \in \mathcal{A}.$$

c) We first show the claim that, if S is a connected regular orientable surface, there exists exactly two orientations on S.

Proof of Claim. Since S is orientable, there exists $N \in \mathcal{N}$. Note that, $-N \in \mathcal{N}$ also. Given any other normal $\tilde{N} \in \mathcal{N}$, we note that $N \cdot \tilde{N} : S \to \{\pm 1\}$ is smooth. Since $\tilde{N}$ is connected, $N \cdot \tilde{N}$ is constant, and thus either $\tilde{N} = N$ or $\tilde{N} = -N$. That is, $\mathcal{N} = \{\pm N\}$. ☐

Since $H = f^{-1}(1)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is the smooth function

$$f(x, y, z) = z^2 - x^2 - y^2,$$

H is an orientable regular surface. However H is not connected, as we can decompose $H = H^+ \sqcup H^-$, where

$$H^+ = \{(x, y, z) \in H : z > 0\}, \quad H^- = \{(x, y, z) \in H : z < 0\}.$$

It is clear $H^\pm$ are non-empty, and each are open in H since $H^+ = H \cap \{z > 0\}$, $H^- = H \cap \{z < 0\}$.

However, each of $H^\pm$ are connected. To see this, we simply note that the projection map from each of them to the $\{z = 0\}$ plane is a diffeomorphism, and $\mathbb{R}^2$ is connected.

Therefore, each of $H^\pm$ has two orientations. Let $\pm N^+$ denote the two possible global unit normal vector fields and H^+, and similarly $\pm N^-$ on H^-. Given any global unit normal vector field N on H, its restriction to each connected component is still a unit normal vector field, and so it must be either

- equal to N^+ on H^+ and equal to N^- on H^-;
- equal to $-N^+$ on H^+ and equal to N^- on H^-;
- equal to N^+ on H^+ and equal to $-N^-$ on H^-;
- equal to $-N^+$ on H^+ and equal to $-N^-$ on H^-.

It follows that H admits exactly **four** orientations.