Solutions 5

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(5.1) Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $X : U \to S$ are isothermal coordinates with
$$[g]_X = \begin{pmatrix} e^{2f} & 0 \\ 0 & e^{2f} \end{pmatrix},$$
for some smooth function $f : U \to \mathbb{R}$.

a) Show that the Christoffel symbols are given by the formula
$$\Gamma^k_{ij} = f_i \delta_{jk} + f_j \delta_{ik} - f_k \delta_{ij}.$$

b) Show that, for a curve $\gamma(t) = X(u_1(t), u_2(t))$ inside of S, that the parallel transport equations for a smooth vector field $W = w_1(t)X_1(t) + w_2(t)X_2(t)$ along γ are the system of equations
$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} + \begin{pmatrix} f_1 u_1'(t) + f_2 u_2'(t) & f_2 u_1'(t) - f_1 u_2'(t) \\ f_1 u_2'(t) - f_2 u_1'(t) & f_1 u_1'(t) + f_2 u_2'(t) \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = 0$$

c) Show that the Gaussian curvature is given by the formula
$$K = -e^{-2f} \Delta f.$$

Solution (5.1)

a) Since $g_{ij} = e^{2f}\delta_{ij}$ and $g^{ij} = e^{-2f}\delta_{ij}$, the intrinsic formula for the Christoffel symbols becomes
$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\
&= \frac{1}{2} e^{-2f} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) \\
&= \frac{1}{2} e^{-2f} \left(\partial_i (e^{2f}) \delta_{jk} + \partial_j (e^{2f}) \delta_{ik} - \partial_k (e^{2f}) \delta_{ij} \right) \\
&= f_i \delta_{jk} + f_j \delta_{ik} - f_k \delta_{ij}.
\end{aligned}$$

b) The local system of equations for parallel transport can be written in matrix form as
$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} + \begin{pmatrix} \Gamma^1_{11} u_1'(t) + \Gamma^1_{12} u_2'(t) & \Gamma^1_{21} u_1'(t) + \Gamma^1_{22} u_2'(t) \\ \Gamma^2_{11} u_1'(t) + \Gamma^2_{12} u_2'(t) & \Gamma^2_{21} u_1'(t) + \Gamma^2_{22} u_2'(t) \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = 0$$

Note that
$$\Gamma^1_{11} = f_1, \quad \Gamma^1_{12} = \Gamma^1_{21} = f_2, \quad \Gamma^1_{22} = -f_1,$$
$$\Gamma^2_{11} = -f_2, \quad \Gamma^2_{12} = \Gamma^2_{21} = f_1, \quad \Gamma^2_{22} = f_2,$$

which substituting into the above gives the desired result.

c) Using the intrinsic formula for the Gaussian curvature we have

$$\begin{aligned} K &= \frac{1}{2} g^{ij} \left(\partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^q \Gamma_{kq}^k - \Gamma_{ik}^q \Gamma_{jq}^k \right) \\ &= \frac{1}{2} e^{-2f} \left(\partial_k \Gamma_{ii}^k - \partial_i \Gamma_{ik}^k + \Gamma_{ii}^q \Gamma_{kq}^k - \Gamma_{ii}^q \Gamma_{iq}^k \right). \end{aligned}$$

Note that

$$\begin{aligned} \partial_k \Gamma_{ii}^k &= \partial_k (2 f_i \delta_{ik} - f_k) = \partial_k (f_k - f_k) = 0, \\ \Gamma_{ii}^q \Gamma_{kq}^q &= (2 f_i \delta_{iq} - f_q) f_q = (f_q - f_q) f_q = 0, \\ \Gamma_{ik}^q \Gamma_{iq}^k &= (f_i \delta_{kq} + f_k \delta_{iq} - f_q \delta_{ik})(f_i \delta_{kq} - f_k \delta_{iq} + f_q \delta_{ik}) \\ &= f_i^2 \delta_{kq} - f_k^2 \delta_{iq} - f_q^2 \delta_{ik} + 2 f_k f_q \delta_{iq} \delta_{ik} \\ &= 2 f_i^2 - 2 f_k^2 - 2 f_q^2 + 2 f_k^2 = 0. \end{aligned}$$

Thus

$$K = -\frac{1}{2} e^{-2f} \partial_i \Gamma_{ik}^k = -\frac{1}{2} e^{-2f} 2 f_{ii} = -e^{-2f} \Delta f.$$

(5.2) Suppose $\gamma : [0,1] \to S$ is a continuous curve in a regular surface S. We further suppose that γ is piecewise smooth and regular. That is, for some finite set of times $0 =: t_0 < t_1 < \ldots < t_k < t_{k+1} := 1$, the curve $\gamma|_{(t_i, t_{i+1})}$ is a smooth regular curve for $i \in \{0, \ldots, k\}$.

In the lectures, we showed that given any tangent vector $v \in T_{\gamma(0)} \mathbb{S}^2$, there exists a unique continuous vector field $W(t)$ along γ with $W(0) = v$, such that $W(t)$ is parallel for times $t \in (0,1) \setminus \{t_1, \ldots, t_k\}$. We say that the **parallel transport along** γ of v from $\gamma(0)$ to $\gamma(1)$ is the vector $W(1) \in T_{\gamma(1)} S$, which we denote by $P_\gamma(v)$.

a) Show that parallel transport along γ defines a linear isomorphism

$$P_\gamma : T_{\gamma(0)} S \to T_{\gamma(1)} S.$$

Hint: consider traversing the curve in the opposite direction
$\overline{\gamma}(t) := \gamma(1-t)$.

b) Recall, stereographic coordinates on the sphere are isothermal coordinates $X : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,-1)\}$, with first fundamental form

$$[g]_X = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}.$$

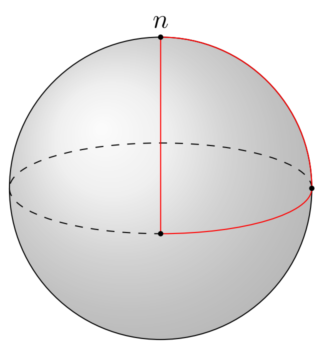
Using your answer to (5.1) part b), show that along the image of straight line $\gamma(t) = X(\cos\theta \cdot t, \sin\theta \cdot t)$ inside the sphere, for any fixed $\theta \in \mathbb{R}$, the parallel transport equations for a vector field $W = w_1(t) X_1(t) + w_2(t) X_2(t)$ along γ is the system of equations

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{2t}{1+t^2} & 0 \\ 0 & \frac{2t}{1+t^2} \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.$$

c) Again, using your answer to (5.1) part b), show that along the image of the unit circle $\gamma(t) = X(\cos t, \sin t)$ inside the sphere, the parallel transport equations for a vector field $W = w_1(t)X_1(t) + w_2(t)X_2(t)$ along γ is the system of equations

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.$$

d) Consider the geodesic triangle γ in the sphere which starts at the north pole $n \in \mathbb{S}^2$, moves down to the equator, traverses a quarter of the way around the equator, and then heads back up to the north pole (see the curve in red below).



Using your answers to parts b) and c), show that the parallel transport along γ defines a map $P_\gamma : T_n\mathbb{S}^2 \to T_n\mathbb{S}^2$ which corresponds to a rotation by $\frac{\pi}{2}$ radians.

Solution (5.2)

a) Fix $u_1, u_2 \in T_{\gamma(0)}S$ and $\lambda \in \mathbb{R}$. Let W_1, W_2 denote the unique parallel vector fields along γ with $W_i(0) = u_i$. Consider the new smooth vector field $W := \lambda W_1 + W_2$ along γ. We note that

$$\frac{DW}{ds} = [(\lambda W_1 + W_2)'(s)]^T = \lambda[W_1'(s)]^T + [W_2'(s)]^T = \lambda\frac{DW_1}{ds} + \frac{DW_2}{ds} = 0.$$

In particular, W is parallel along γ with $W(0) = \lambda u_1 + u_2$, and thus

$$P_\gamma(\lambda u_1 + u_2) = W(1) = \lambda W_1(1) + W_2(1) = \lambda P_\gamma(u_1) + P_\gamma(u_2),$$

i.e. P_γ is a linear map. To show it is a linear isomorphism, we just have to construct its inverse. Following the hint, we consider $P_{\overline{\gamma}} : T_{\gamma(1)}S \to T_{\gamma(0)}S$. Note that for any parallel vector field W along γ, the vector field $\overline{W}(t) := W(1-t)$ is parallel along $\overline{\gamma}$. Therefore

$$P_{\overline{\gamma}} \circ P_\gamma(W(0)) = P_{\overline{\gamma}}(W(1)) = P_{\overline{\gamma}}(\overline{W}(0)) = \overline{W}(1) = W(0),$$

and similarly $P_\gamma \circ P_{\overline{\gamma}} = \mathrm{id}_{T_{\gamma(1)}S}$.

b) Writing g in the form as given in Question 5.1, we see that

$$f(x,y) = \log 2 - \log(1 + x^2 + y^2),$$

and hence

$$f_1 = \frac{-2x}{1 + x^2 + y^2}, \quad f_2 = \frac{-2y}{1 + x^2 + y^2}.$$

For this choice of γ we see that $u_1'(t) = \cos\theta$, $u_2'(t) = \sin\theta$ and

$$f_1(t) = \frac{-2t\cos\theta}{1 + t^2}, \quad f_2(t) = \frac{-2t\sin\theta}{1 + t^2},$$

which when plugged into the result of Question 5.1 part b) yields

$$\begin{aligned}
\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} &= \begin{pmatrix} \frac{2t\cos\theta}{1+t^2}\cos\theta + \frac{2t\sin\theta}{1+t^2}\sin\theta & \frac{2t\sin\theta}{1+t^2}\cos\theta - \frac{2t\cos\theta}{1+t^2}\sin\theta \\ \frac{2t\cos\theta}{1+t^2}\cos\theta - \frac{2t\sin\theta}{1+t^2}\cos\theta & \frac{2t\cos\theta}{1+t^2}\cos\theta + \frac{2t\sin\theta}{1+t^2}\cos\theta \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{2t}{1+t^2} & 0 \\ 0 & \frac{2t}{1+t^2} \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.
\end{aligned}$$

c) For this choice of γ we see that $u_1'(t) = -\sin t$, $u_2'(t) = \cos t$ and

$$f_1(t) = -\cos t, \quad f_2(t) = -\sin t,$$

which when plugged into the result of Question 5.1 part b) yields

$$\begin{aligned}
\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} &= \begin{pmatrix} -\cos t\sin t + \cos t\sin t & -\sin^2 t - \cos^2 t \\ \cos^2 t + \sin^2 t & -\cos t\sin t + \sin t\cos t \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.
\end{aligned}$$

d) Solving the system of equations from part b), we see that

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = (1 + t^2) \cdot \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix},$$

and solving the system of equations from part c), we see that

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}.$$

Letting $\gamma_1(t) = (t, 0)$, $\gamma_2(t) = (\cos t, \sin t)$, and $\gamma_3(t) = (0, t)$, within the stereographic coordinates from part b) γ corresponds to the curve

$$t \mapsto \begin{cases} \gamma_1(t) & : t \in [0, 1], \\ \gamma_2(t - 1) & : t \in [1, \frac{\pi}{2} + 1], \\ \gamma_3(\frac{\pi}{2} + 1 - t) & : t \in [\frac{\pi}{2} + 1, \frac{\pi}{2} + 2], \end{cases}$$

Let $v := (v_1, v_2) \in T_n\mathbb{S}^2$ be any vector in the tangent space. and hence if $w_i(0) = v_i$, we have that

$$\begin{pmatrix} w_1(1) \\ w_2(1) \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}, \quad \begin{pmatrix} w_1(1 + \frac{\pi}{2}) \\ w_2(1 + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -2v_2 \\ 2v_1 \end{pmatrix}, \quad \begin{pmatrix} w_1(2 + \frac{\pi}{2}) \\ w_2(2 + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}.$$

Therefore, $P_\gamma(v_1, v_2) = (-v_2, v_1)$, which is a rotation about 90 degrees.

(5.3) Suppose γ is a smooth closed simple curve inside $\mathbb{S}^2$. By the Jordan curve theorem, γ splits $\mathbb{S}^2$ into two regions. Show that these two regions have equal area if and only if

$$\int_\gamma \kappa_G \, ds = 0.$$

Solution (5.3) Since γ is a smooth closed simple curve in the sphere, there exists $p \in \mathbb{S}^2$ not in its trace, and hence by taking stereographic projection, we may assume γ is contained within a single (isothermal) coordinate chart. Then, by the local Gauss Bonnet theorem, we have that

$$\int_\Omega K \, dA + \int_\gamma \kappa_G \, ds = 2\pi,$$

where Ω denotes the compact region in the coordinate chart bounded by γ. Since $K \equiv 1$, it follows that

$$\int_\gamma \kappa_G \, ds = 2\pi - \int_\Omega dA,$$

and therefore the total geodesic curvature is zero iff the area of Ω is 2π, which is precisely half the area of the sphere.