Solutions 4

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(4.1) Consider a regular surface given by the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

a) Find a formula for the Gaussian curvature K and the mean curvature H on the surface in terms of f and its partial derivatives.

b) Find an example of a regular surface S with a planar point $p \in S$ for which every neighbourhood $p \in V \subseteq S$ contains points laying on both sides of the hyperplane $p + T_pS$.

Hint: Consider degree three homogeneous polynomials
$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

c) Find an example of a regular surface S with vanishing Gaussian curvature $K \equiv 0$, but with points $p, q \in S$ such that

$$H(q) < 0 < H(p).$$

Solution (4.1)

a) Using the global coordinate chart $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$, we find that

$$g = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}, \quad h = \frac{1}{\sqrt{1 + f_1^2 + f_2^2}} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

It follows that

$$K = \frac{f_{11} f_{22} - f_{12}^2}{(1 + f_1^2 + f_2^2)^{\frac{3}{2}}}, \quad H = \frac{f_{22}(1 + f_1^2) + f_{11}(1 + f_2^2) - 2f_{12} f_1 f_2}{2(1 + f_1^2 + f_2^2)^{\frac{3}{2}}}.$$

b) Set $f(x, y) = x^3 - 3x^2y$ and $p = (0, 0, 0)$. Since f is homogeneous degree three, p is planar. Define the sequence of points $q_n = (\frac{1}{n}, \frac{1}{n})$, $p_n = (\frac{1}{n}, 0)$. Both $p_n, q_n \to (0, 0)$ as $n \to \infty$, and

$$f(q_n) = -\frac{2}{n^3} < 0 < \frac{1}{n^3} = f(p_n).$$

c) Set $f(x, y) = \cos x$, so that $K \equiv 0$, but

$$H = \frac{f_{11}}{2(1 + f_1^2)^{\frac{3}{2}}} = \frac{-\cos x}{2(1 + \sin^2 x)^{\frac{3}{2}}}.$$

Take $q = (0, 0, 1)$ and $p = (\pi, 0, -1)$.

(4.2) Let $S \subseteq \mathbb{R}^3$ be a compact orientable surface with Gauss map $N : S \to \mathbb{S}^2$. For each $p \in S$ and $r > 0$, we define the open subset of S

$$B_S(p, r) := \{x \in S : \|x - p\| < r\} = B(p, r) \cap S.$$

Consider the smooth function $\Psi : S \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$\Psi(p, t) := p + t \cdot N_p, \quad \forall (p, t) \in S \times \mathbb{R}.$$

a) Show that for every point $p \in S$, there exists some $\epsilon_p > 0$ such that Ψ restricted to $B_S(p, \epsilon_p) \times (-\epsilon_p, \epsilon_p)$ is smooth diffeomorphism onto its image.

b) Fix $p, q \in S$ and let $|t| < \epsilon_p$ and $|s| < \epsilon_q$. If $\Psi(p, t) = \Psi(q, s)$, show that

$$\|p - q\| \leq \epsilon_p + \epsilon_q.$$

c) Show that there exists some $\epsilon > 0$ such that Ψ restricted to $S \times (-\epsilon, \epsilon)$ is a smooth diffeomorphism onto its image $N_\epsilon S \subseteq \mathbb{R}^3$.

d) Given a smooth function $f : S \to (-\epsilon, \epsilon)$, we define the corresponding section of the normal bundle to be the function $\sigma_f : S \to N_\epsilon S$ given by

$$\sigma_f(p) := p + f(p) N_p, \quad \forall p \in S.$$

Show that $\sigma_f(S)$ is homeomorphic to S.

Solution (4.2)

a) In local coordinates $X : U \to S$ about a point p, with $X(0, 0) = p$, we have the smooth function

$$\hat{\Psi}(u, v, t) := \Psi(X(u, v), t) = X(u, v) + t \cdot N_{(u,v)}.$$

Differentiating at the point $(0, 0, 0) \in U \times \mathbb{R}$ we have

$$\hat{\Psi}_u(0, 0, 0) = X_u(0, 0), \quad \hat{\Psi}_v(0, 0, 0) = X_v(0, 0), \quad \hat{\Psi}_t(0, 0, 0) = N_{(0,0)},$$

which are linearly independent, and so $d\hat{\Psi}(0, 0, 0)$ is invertible. Thus, by the Inverse Function Theorem, there exists an open subset $(0, 0, 0) \in W \subseteq U \times \mathbb{R}$ such that $\hat{\Psi} \llcorner W$ is a diffeomorphism onto its image. Since W is open, after possibly shrinking W we may assume it has the form

$$W = V \times I,$$

where $(0, 0) \in V \subseteq U$ is open and $0 \in I \subseteq \mathbb{R}$ is an open interval. Since X is a diffeomorphism, it follows that Ψ restricted to $X(V) \times I$ is a diffeomorphism. As $p \in X(V)$ is an open subset of S, the result follows.

b) $\Psi(p, t) = \Psi(q, s) \iff p + tN_p = q + sN_q$. Rearranging and using the triangle inequality

$$\|p - q\| = \|tN_p - sN_q\| \leq |t| \, \|N_p\| + |s| \, \|N_q\| = |t| + |s| \leq \epsilon_p + \epsilon_q.$$

c) By part a), we can cover S by open subsets $B_S(p, \frac{\epsilon_p}{2})$. Since S is compact, we may restrict to a finite subcover

$$S \subseteq \bigcup_{i=1}^{N} B_S(p_i, \frac{\epsilon_{p_i}}{2})$$

Set $\epsilon := \frac{1}{2}\min\{\epsilon_{p_1}, \ldots, \epsilon_{p_N}\} > 0$. By part a), the functions Ψ piece together to form a smooth map

$$\Psi : S \times (-\epsilon, \epsilon) \to \mathbb{R}^3.$$

If Ψ is injective, then its inverse is well-defined and agrees with the local inverse given in part a) which is smooth, so we are done.

To show Ψ is injective, suppose $\Psi(p,t) = \Psi(q,s)$ for some $(p,t),(q,s) \in S \times (-\epsilon, \epsilon)$. We know that $p \in B_S(p_i, \epsilon_i)$ for some $i \in \{1, \ldots, N\}$. By part b) we must have

$$\|p - q\| \leq 2\epsilon < \epsilon_{p_i},$$

and hence $q \in B_S(p_i, \epsilon_i)$ also. But then by part a), $(p,t) = (q,s)$ and Ψ is injective.

d) Since

$$\sigma_f(p) = \Psi(p, f(p)), \quad \forall p \in S,$$

σ_f is a smooth bijective map onto its image. It remains to show σ_f^{-1} is continuous.

Let $\pi : S \times (-\epsilon, \epsilon) \to S$ denote the projection map onto its first component. By the composition of smooth functions, the projection map $\pi \circ \Psi^{-1} : N_\epsilon S \to S$ is smooth. In particular, its restriction to $\sigma_f(S)$ is continuous. But this is precisely σ_f^{-1}.

(4.3) Recall from §4.3 of the lecture notes that coordinates $X : U \to S$ are called *isothermal coordinates* if there is a smooth function $\lambda : U \to \mathbb{R}$ such that the first fundamental form

$$[g_{(u,v)}]_X = \begin{pmatrix} \lambda^2(u,v) & 0 \\ 0 & \lambda^2(u,v) \end{pmatrix}, \quad \forall (u,v) \in U.$$

a) Show that $\langle X_{uu}, X_u \rangle = \lambda \cdot \lambda_u$.

b) Show that ΔX is parallel to the normal vector N.

c) Find a formula relating ΔX with the mean curvature H, and conclude that H vanishes on $X(U)$ iff X is harmonic on U.

Solution (4.3)

a)
$$\langle X_{uu}, X_u \rangle = \frac{1}{2}\partial_u \cdot \langle X_u, X_u \rangle = \lambda \cdot \lambda_u$$

b)
$$\begin{aligned}\langle X_{vv}, X_u \rangle &= \partial_v \cdot \langle X_v, X_u \rangle - \langle X_v, X_{vu} \rangle \\ &= -\frac{1}{2}\partial_u \cdot \langle X_v, X_v \rangle = -\lambda \cdot \lambda_u,\end{aligned}$$

and hence

$$\langle \Delta X, X_u \rangle = \langle X_{uu} + X_{vv}, X_u \rangle = \lambda\lambda_u - \lambda\lambda_u = 0.$$

Similarly, $\langle \Delta X, X_v \rangle = 0$, and so $\Delta X(p) \perp T_pS$ and hence parallel to N_p.

c) From the notes we have that

$$H(p) = \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - g_{12}^2)} = \frac{h_{11} + h_{22}}{2\lambda^2}.$$

Since $h_{11} + h_{22} = \langle X_{uu} + X_{vv}, N \rangle = \langle \Delta X, N \rangle$, we find that

$$\Delta X(p) = \langle \Delta X(p), N_p \rangle\, N_p = 2\lambda^2 H(p) \cdot N_p.$$

Since the normal vector is always non-zero, and since $\lambda^2 \neq 0$ (as g is non-degenerate), we conclude that X is harmonic iff $H \equiv 0$.