Homework 2

Solutions

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(2.1) Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $M \subseteq S$, with $M \neq \emptyset$. Show that M is a regular surface if and only if M is an open subset of S.

Recall, M is an open subset of S if there exists some open subset $V \subseteq \mathbb{R}^3$ such that $M = S \cap V$.

Solution (2.1) Assume $M = S \cap V$ for some open set V. For every coordinate chart $X : U \to S$, we see that $M \cap X(U) = V \cap X(U)$ is an open subset of $X(U)$, and hence $U \cap X^{-1}(M)$ is an open subset of $\mathbb{R}^2$. Therefore, we can consider the restrictions $X : U \cap X^{-1}(M) \to M$. Properties (i) and (ii) are still true for these restrictions, and so we have well-defined coordinate charts on M, and hence M is a regular surface.

Conversely, if M is a regular surface, fix $p \in M \subseteq S$, and let $X_M : U \to M$ be a coordinate chart on a neighbourhood of p in M. Similarly, let $X_S : \tilde{U} \to S$ be a coordinate chart on a neighbourhood of p in S. Choosing $W = X_M(U) \cap X_S(\tilde{U}) \ni p$, we have that

$$X_S^{-1} \circ X_M : X_M^{-1}(W) \to X_S^{-1}(W),$$

is a homeomorphism, and therefore

$$W = X_M(X_M^{-1}(W)) = X_S \circ (X_S^{-1} \circ X_M)(X_M^{-1}(W)),$$

is an open subset of S. It follows that M is open in S.

(2.2) Given a regular surface $S \subseteq \mathbb{R}^3$, define

$$\text{Diff}(S) := \{f : S \to S : f \text{ is a smooth diffeomorphism}\}.$$

a) Show that $\text{Diff}(S)$ forms a group where the binary operation is given by the composition of functions. We call $\text{Diff}(S)$ the diffeomorphism group of S.

b) Let $A : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map

$$A(x, y, z) = (-x, -y, -z), \quad \forall (x, y, z) \in \mathbb{S}^2.$$

Show that $A \in \text{Diff}(\mathbb{S}^2)$, and that A has order 2 in the group.

c) For each $n \in \mathbb{N}$, find a diffeomorphism $f \in \text{Diff}(\mathbb{S}^2)$ with order n in the group.

d) Find a diffeomorphism $f \in \text{Diff}(\mathbb{S}^2)$ with infinite order.

Hint: $O(3) \subseteq \text{Diff}(\mathbb{S}^2)$.

Solution (2.2)

a) To begin, the identity map $\mathrm{id} : \mathbb{R}^3 \to \mathbb{R}^3$ is smooth, and so by the composition of smooth functions, $\mathrm{id} \circ X : U \to \mathbb{R}^3$ is smooth for any coordinate chart $X : U \to S$. Therefore, the restriction of the identity map $S \to S$ is a smooth bijection. Moreover, its inverse is itself, so it belongs to the set $\mathrm{id} \in \text{Diff}(S)$.

By the definition of a smooth diffeomorphism, if $f \in \text{Diff}(S)$, then $f^{-1} \in \text{Diff}(S)$. Given $f, g \in \text{Diff}(S)$, the composition $f \circ g$ is smooth with smooth inverse $g^{-1} \circ f^{-1}$. Thus, $f \circ g \in \text{Diff}(S)$.

b) Again, since the map from $\mathbb{R}^3$ to itself given by $(x, y, z) \mapsto (-x, -y, -z)$ is smooth, by the composition of smooth functions, the antipodal map given by restriction to the sphere $A : \mathbb{S}^2 \to \mathbb{S}^2$ is smooth. Moreover, as $A^{-1} = A$, A^{-1} is smooth and $A \in \text{Diff}(\mathbb{S}^2)$.
Note that $A \circ A = \mathrm{id}$, and thus A has order 2.

c) Fix $\theta \in \mathbb{R}$ and consider the rotation about the z-axis in $\mathbb{R}^3$, $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$,

$$R_\theta(x, y, z) = (x \cos\theta + y \sin\theta, y \cos\theta - x \sin\theta, z), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

Since R_θ is smooth, we again apply the composition of smooth functions to see that its restriction to the sphere is smooth. Note that

$$\|R_\theta(x, y, z)\|^2 = (x\cos\theta + y\sin\theta)^2 + (y\cos\theta - x\sin\theta)^2 + z^2 = x^2 + y^2 + z^2 = \|(x, y, z)\|^2,$$

and therefore $R_\theta : \mathbb{S}^2 \to \mathbb{S}^2$ is a smooth map. Finally, since $R_\theta \circ R_\sigma = R_{\theta+\sigma}$ and $R_{2\pi k} = \mathrm{id}$ for any $k \in \mathbb{Z}$, we see that $R_\theta^{-1} = R_{-\theta}$ is smooth, and hence $R_\theta \in \text{Diff}(\mathbb{S}^2)$.

Choosing $f := R_{\frac{2\pi}{n}}$, we see that

$$f^j = \underbrace{f \circ \cdots \circ f}_{j \text{ times}} = R_{\frac{2\pi j}{n}},$$

which is the identity map iff n divides j. Thus, f has order n.

d) if we instead choose $f = R_{2\pi\theta}$ for some θ irrational. Then, for any $j \in \mathbb{N}$, we have

$$f^j = R_{2\pi j\theta} \neq \mathrm{id},$$

and hence f has infinite order.

(2.3) Let $S_1, S_2, S_3 \subseteq \mathbb{R}^3$ be regular surfaces, and let $f : S_1 \to S_2$, and $g : S_2 \to S_3$ be smooth functions.

Show that the composition $g \circ f : S_1 \to S_3$ is smooth, and that

$$d(g \circ f)(p) = dg(f(p)) \circ df(p),$$

for any $p \in S_1$.

Solution (2.3) Fix $p \in S_1$, and let $X : U \to S_1$, $Y : V :\to S_2$, $Z : W \to S_3$ be coordinate charts about the points p, $f(p)$ and $g(f(p))$ respectively. Then the composition

$$Z^{-1} \circ g \circ f \circ X = (Z^{-1} \circ g \circ Y) \circ (Y^{-1} \circ f \circ X),$$

is the composition of smooth functions, and hence smooth.

Fix $v \in T_pM$ and let $\gamma : (-\epsilon, \epsilon) \to S_1$ be a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = v$. From the definition of derivatives, it follows that

$$\begin{aligned} dg(f(p)) \circ df(p) \cdot v &= dg(f(p)) \cdot (f \circ \gamma)'(0) \\ &= (g \circ f \circ \gamma)'(0) \\ &= d(g \circ f)(p) \cdot v. \end{aligned}$$

(2.4) Suppose $S \subseteq \mathbb{R}^3$ is a connected regular surface and $f : S \to \mathbb{R}$ is a smooth function. Show that f is a constant function if and only if $df(p) = 0$ as a linear map, for every $p \in S$.

Hint: as $S \subseteq \mathbb{R}^3$ is a connected regular surface, you may assume that, for any two points $x, y \in S$, there exists $\epsilon > 0$, and a smooth regular curve $\gamma : (-\epsilon, 1 + \epsilon) \to S$, such that $\gamma(0) = x$ and $\gamma(1) = y$.

Solution (2.4) Suppose $f \equiv c$. Then for any curve $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$, we have

$$df(p) \cdot \gamma'(0) = (f \circ \gamma)'(0) = 0,$$

and hence $df(p) = 0$. Conversely, suppose $df(p) = 0$ everywhere. Fix $x, y \in S$ and choose γ as in the hint. By the fundamental theorem of calculus, we see that

$$\begin{aligned} f(y) - f(x) &= f \circ \gamma(1) - f \circ \gamma(0) \\ &= \int_0^1 (f \circ \gamma)'(t) dt \\ &= \int_0^1 \underbrace{df(\gamma(t))}_{=0} \cdot \gamma'(t) dt = 0, \end{aligned}$$

and hence f is constant.