Homework 4

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Friday 15th November 2024. Please let me know if any of the problems are unclear or have typos.

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(4.1) Consider a regular surface given by the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

a) Find a formula for the Gaussian curvature K and the mean curvature H on the surface in terms of f and its partial derivatives.

b) Find an example of a regular surface S with a planar point $p \in S$ for which every neighbourhood $p \in V \subseteq S$ contains points laying on both sides of the hyperplane $p + T_pS$.

Hint: Consider degree three homogeneous polynomials
$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

c) Find an example of a regular surface S with vanishing Gaussian curvature $K \equiv 0$, but with points $p, q \in S$ such that

$$H(q) < 0 < H(p).$$

(4.2) Let $S \subseteq \mathbb{R}^3$ be a compact orientable surface with Gauss map $N : S \to \mathbb{S}^2$. For each $p \in S$ and $r > 0$, we define the open subset of S

$$B_S(p, r) := \{x \in S : \|x - p\| < r\} = B(p, r) \cap S.$$

Consider the smooth function $\Psi : S \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$\Psi(p, t) := p + t \cdot N_p, \quad \forall (p, t) \in S \times \mathbb{R}.$$

a) Show that for every point $p \in S$, there exists some $\epsilon_p > 0$ such that Ψ restricted to $B_S(p, \epsilon_p) \times (-\epsilon_p, \epsilon_p)$ is smooth diffeomorphism onto its image.

b) Fix $p, q \in S$ and let $|t| < \epsilon_p$ and $|s| < \epsilon_q$. If $\Psi(p, t) = \Psi(q, s)$, show that

$$\|p - q\| \leq \epsilon_p + \epsilon_q.$$

c) Show that there exists some $\epsilon > 0$ such that Ψ restricted to $S \times (-\epsilon, \epsilon)$ is a smooth diffeomorphism onto its image $N_\epsilon S \subseteq \mathbb{R}^3$.

d) Given a smooth function $f : S \to (-\epsilon, \epsilon)$, we define the corresponding section of the normal bundle to be the function $\sigma_f : S \to N_\epsilon S$ given by

$$\sigma_f(p) := p + f(p)N_p, \quad \forall p \in S.$$

Show that $\sigma_f(S)$ is homeomorphic to S.

(4.3) Recall from §4.3 of the lecture notes that coordinates $X : U \to S$ are called *isothermal coordinates* if there is a smooth function $\lambda : U \to \mathbb{R}$ such that the first fundamental form

$$[g_{(u,v)}]_X = \begin{pmatrix} \lambda^2(u,v) & 0 \\ 0 & \lambda^2(u,v) \end{pmatrix}, \quad \forall (u,v) \in U.$$

a) Show that $\langle X_{uu}, X_u \rangle = \lambda \cdot \lambda_u$.

b) Show that ΔX is parallel to the normal vector N.

c) Find a formula relating ΔX with the mean curvature H, and conclude that H vanishes on $X(U)$ iff X is harmonic on U.