Homework 5

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Friday 29th November 2024. Please let me know if any of the problems are unclear or have typos.

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(5.1) Suppose $S \subseteq \mathbb{R}^3$ is a regular surface and $X : U \to S$ are isothermal coordinates with

$$[g]_X = \begin{pmatrix} e^{2f} & 0 \\ 0 & e^{2f} \end{pmatrix},$$

for some smooth function $f : U \to \mathbb{R}$.

a) Show that the Christoffel symbols are given by the formula

$$\Gamma_{ij}^k = f_i \delta_{jk} + f_j \delta_{ik} - f_k \delta_{ij}.$$

b) Show that, for a curve $\gamma(t) = X(u_1(t), u_2(t))$ inside of S, that the parallel transport equations for a smooth vector field $W = w_1(t)X_1(t) + w_2(t)X_2(t)$ along γ are the system of equations

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} + \begin{pmatrix} f_1 u_1'(t) + f_2 u_2'(t) & f_2 u_1'(t) - f_1 u_2'(t) \\ f_1 u_2'(t) - f_2 u_1'(t) & f_1 u_1'(t) + f_2 u_2'(t) \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = 0$$

c) Show that the Gaussian curvature is given by the formula

$$K = -e^{-2f} \Delta f.$$

(5.2) Suppose $\gamma : [0,1] \to S$ is a continuous curve in a regular surface S. We further suppose that γ is piecewise smooth and regular. That is, for some finite set of times $0 =: t_0 < t_1 < \ldots < t_k < t_{k+1} := 1$, the curve $\gamma|_{(t_i, t_{i+1})}$ is a smooth regular curve for $i \in \{0, \ldots, k\}$.

In the lectures, we showed that given any tangent vector $v \in T_{\gamma(0)}\mathbb{S}^2$, there exists a unique continuous vector field $W(t)$ along γ with $W(0) = v$, such that $W(t)$ is parallel for times $t \in (0,1) \setminus \{t_1, \ldots, t_k\}$. We say that the **parallel transport along** γ of v from $\gamma(0)$ to $\gamma(1)$ is the vector $W(1) \in T_{\gamma(1)}S$, which we denote by $P_\gamma(v)$.

a) Show that parallel transport along γ defines a linear isomorphism

$$P_\gamma : T_{\gamma(0)}S \to T_{\gamma(1)}S.$$

Hint: consider traversing the curve in the opposite direction
$\overline{\gamma}(t) := \gamma(1-t)$.

b) Recall, stereographic coordinates on the sphere are isothermal coordinates $X : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,-1)\}$, with first fundamental form

$$[g]_X = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}.$$

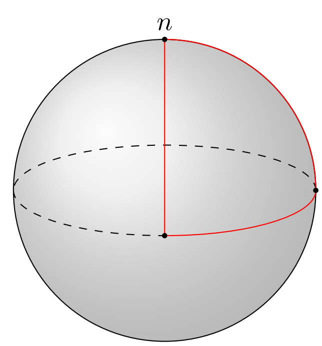
Using your answer to (5.1) part b), show that along the image of straight line $\gamma(t) = X(\cos\theta \cdot t, \sin\theta \cdot t)$ inside the sphere, for any fixed $\theta \in \mathbb{R}$, the parallel transport equations for a vector field $W = w_1(t)X_1(t) + w_2(t)X_2(t)$ along γ is the system of equations

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{2t}{1+t^2} & 0 \\ 0 & \frac{2t}{1+t^2} \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.$$

c) Again, using your answer to (5.1) part b), show that along the image of the unit circle $\gamma(t) = X(\cos t, \sin t)$ inside the sphere, the parallel transport equations for a vector field $W = w_1(t)X_1(t) + w_2(t)X_2(t)$ along γ is the system of equations

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}.$$

d) Consider the geodesic triangle γ in the sphere which starts at the north pole $n \in \mathbb{S}^2$, moves down to the equator, traverses a quarter of the way around the equator, and then heads back up to the north pole (see the curve in red below).



Using your answers to parts b) and c), show that the parallel transport along γ defines a map $P_\gamma : T_n\mathbb{S}^2 \to T_n\mathbb{S}^2$ which corresponds to a rotation by $\frac{\pi}{2}$ radians.

(5.3) Suppose γ is a smooth closed simple curve inside $\mathbb{S}^2$. By the Jordan curve theorem, γ splits $\mathbb{S}^2$ into two regions. Show that these two regions have equal area if and only if

$$\int_\gamma \kappa_G \, ds = 0.$$