Homework 3

Please submit, through Blackboard, solutions to *all* of the following problems. The deadline for submissions is 18:00 on Friday 18th October 2024. Please let me know if any of the problems are unclear or have typos.

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(3.1) Given a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, for each $\theta \in \mathbb{R}$, consider the curve $\gamma_\theta : \mathbb{R} \to \text{Graph}(f)$, defined by

$$\gamma_\theta(t) = (t\cos\theta, t\sin\theta, f(t\cos\theta, t\sin\theta)), \quad \forall t \in \mathbb{R}.$$

Recall from lectures that the first fundamental form with respect to the coordinates $X : \mathbb{R}^2 \to \text{Graph}(f)$, $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$, is given by

$$g = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}.$$

a) For each $\theta \in \mathbb{R}$, show that the length of the curve restricted to the interval $[-1, 1]$ is given by

$$L(\gamma_\theta|_{[-1,1]}) = \int_{-1}^{1} \sqrt{1 + (f_1\cos\theta + f_2\sin\theta)^2}dt.$$

b) In the case $f(x, y) = x^2 - y^2$, find the values of θ which minimise the length $L(\gamma_\theta|_{[-1,1]})$.

c) In the case $f(x, y) = e^{xy}$, find the values of θ which minimise the length $L(\gamma_\theta|_{[-1,1]})$.

(3.2) Let $X : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\}$ denote the stereographic coordinates

$$X(u, v) := \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

For each $r > 0$, let

$$\Omega_r := X\left(\{(u, v) \in \mathbb{R}^2 : \sqrt{u^2 + v^2} \le r\}\right) \subseteq \mathbb{S}^2.$$

That is, Ω_r is the image under stereographic coordinates of the the closed disk of radius r centred at the origin in the plane.

a) Calculate the area of the region Ω_r.

b) Find a sequence of numbers $r_n \uparrow \infty$ such that the ratio of the area of Ω_{r_n} to the area of its complement $\overline{\mathbb{S}^2 \setminus \Omega_{r_n}}$ is exactly $n : 1$.

(3.3) Prove that any surface of revolution is an orientable surface.

(3.4) Let S be a regular surface. Given a collection of charts $\{X^i : U_i \to S\}_{i \in I}$ over some index I, we say that they form an **atlas** on S if they cover S

$$\bigcup_{i \in I} X^i(U_i) = S.$$

We say an atlas $\{X^i : U_i \to S\}_{i \in I}$ is **orientable** if the charts define a unique orientation on every tangent space of S. Therefore, S is orientable if and only if S admits an orientable atlas.

We consider the collection of all orientable atlases on our surface S

$$\mathcal{A} = \{\mathfrak{a} = \{X^i : U_i \to S\}_{i \in I} : \mathfrak{a} \text{ is an orientable atlas on } S\}.$$

Given two orientable atlases $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{A}$, their union $\mathfrak{a}_1 \cup \mathfrak{a}_2$ is also an atlas on S. We define the relation $\mathfrak{a}_1 \sim \mathfrak{a}_2$ if $\mathfrak{a}_1 \cup \mathfrak{a}_2 \in \mathcal{A}$, i.e. their union is also an orientable atlas.

a) Show that $\sim$ given above is a well-defined equivalence relation on $\mathcal{A}$. In particular, we can then define the space of orientations on S to be the quotient space

$$Or(S) = \mathcal{A}/\sim .$$

We now consider the space of smooth unit normal vector fields

$$\mathcal{N} := \{N : S \to \mathbb{R}^3 : N \text{ is a smooth with } \|N_p\| = 1,\ N_p \perp T_pS,\ \forall p \in S\}.$$

In the lectures (Lemma 3.17) we showed that $Or(S) \neq \emptyset \iff \mathcal{N} \neq \emptyset$.

b) Find a bijection between the sets $Or(S)$ and $\mathcal{N}$.

c) Consider the hyperboloid

$$H := \{(x, y, z) \in \mathbb{R}^3 : -x^2 - y^2 + z^2 = 1\}.$$

It was shown in lectures that H is a regular surface.

How many distinct orientations does H have? That is, what is the cardinality of the set $Or(H)$? Justify your answer.