MATH 3310 Assignment 3

Due: November 5, 2024

1. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \; and \; \; b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

   (a) Determine whether the Jacobi method converges.

   (b) Using initial approximation $x^0 = (3, 2, 1)^T$, conduct the first two Jacobi iterations.

   (c) Determine whether the Gauss-Seidel method converges.

   (d) Using initial approximation $x^0 = (2, 1, 0)^T$, conduct the first two Gauss-Seidel iterations.

solution:

(a)

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; P = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

so we get

$$M = D^{-1}P = \begin{pmatrix} 0 & -1 & -1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}$$

It can be verified that the spectral radius $\rho(D^{-1}P) \approx 1.1372 > 1$, so the Jacobi method diverges.

(b) $x^1 = D^{-1}Px^0 + D^{-1}b = (-2, -1, -\frac{2}{3})^T$;
$x^2 = D^{-1}Px^1 + D^{-1}b = (\frac{8}{3}, \frac{7}{3}, 2)^T$.

(c)

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}; P = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so we get $\rho(N^{-1}P) = \frac{1}{2} < 1$, so the Gauss-Seidel method converges.

(d) $x^1 = N^{-1}Px^0 + N^{-1}b = (0, 1, \frac{2}{3})^T$;
$x^2 = N^{-1}Px^1 + N^{-1}b = (-\frac{2}{3}, 1, \frac{8}{9})^T$

2. Consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 8 \\ 8 \\ 8 \end{pmatrix}$$

(a) Determine whether the SOR method converges if $\omega = 1.5$.

(b) Using initial approximation $x^{(0)} = (6, 4, 0)^T$, conduct the first two SOR iteration.

solution:

(a)

$$N = \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 2 & 1 & 2 \end{pmatrix}; P = \begin{pmatrix} \frac{-2}{3} & 0 & -2 \\ 0 & \frac{-2}{3} & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

so we get $\rho(N^{-1}P) = 0.5 < 1$, so the SOR method converges if $\omega = 1.5$.

(b) $x^1 = N^{-1}Px^0 + N^{-1}b = (3, 4, -1)^T$;
$x^2 = N^{-1}Px^1 + N^{-1}b = (6, \frac{19}{4}, -\frac{31}{8})^T$

3. Consider solving $Ax = b$ with the following iterative scheme.

$$x_{k+1} = x_k + \alpha(b - Ax_k)$$

where $\alpha > 0$. Assume all eigenvalues λ_i of A are real, and such that $\lambda_{min} \leq \lambda_i \leq \lambda_{max}$, $\lambda_{min}, \lambda_{max}$ are the smallest and largest eigenvalues of A, respectively.

(a) If $\lambda_{min} < 0$ and $\lambda_{max} > 0$, prove that this scheme always diverges for some initial guess.

(b) Assume $\lambda_{min} > 0$, what are the sufficient and necessary conditions on α for the iterative scheme to converge?

(c) Assume $\lambda_{min} > 0$, what is the best value α_{opt} for α, i.e., the value of α minimizing the convergence factor? And, what is the convergence factor in such case?

solution:

(a) The iteration can be rewritten as

$$x_{k+1} = (I - \alpha A)x_k + \alpha b$$

Thus, the iteration matrix is $G_\alpha = I - \alpha A$ and the convergence factor is $\rho(I - \alpha A)$, Then, the eigenvalues μ_i of G_α are such that

$$1 - \alpha\lambda_{max} \leq \mu_i \leq 1 - \alpha\lambda_{min}$$

if $\lambda_{min} < 0$, at least one eigenvalue is larger than 1 and $\rho(G_\alpha) > 1$ for any α. In this case the method will always diverge for some initial guess.

(b) the method converge iff $\rho(G_\alpha) < 1$ iff

$$\begin{aligned} 1 - \alpha\lambda_{min} &< 1 \\ 1 - \alpha\lambda_{max} &> -1 \end{aligned}$$

so $0 < \alpha < \frac{2}{\lambda_{max}}$

(c) $\rho(G_\alpha) = \max\{|1 - \alpha\lambda_{min}|, |1 - \alpha\lambda_{max}|\}$. By drawing the curves of $f_1(\alpha) = |1 - \alpha\lambda_{min}|$ and $f_2(\alpha) = |1 - \alpha\lambda_{max}|$, we know that the best possibleα is reached at the point where the curve $|1 - \alpha\lambda_{max}|$ with positive slope crosses the curve $|1 - \alpha\lambda_{min}|$with negative slope, i.e.

$$-1 + \alpha\lambda_{max} = 1 - \alpha\lambda_{min}$$

. This gives $\alpha_{opt} = \frac{2}{\lambda_{min}+\lambda_{max}}$ and in such case, $\rho_{opt} = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{max}+\lambda_{min}}$.

4. Consider an $n \times n$ tridiagonal matrix of the form

$$T_\alpha = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \alpha \end{pmatrix},$$

where α is a real parameter.

(a) Verify that the eigenvalues of T_α are given by

$$\lambda_j = \alpha - 2\cos(j\theta), \quad j = 1, \dots, n,$$

where

$$\theta = \frac{\pi}{n+1},$$

and that an eigenvector associated with each λ_j is

$$q_j = [\sin(j\theta), \sin(2j\theta), \dots, \sin(nj\theta)]^T.$$

Under what condition on α does this matrix become positive definite?

(b) Let $\alpha = 2$.

i. Will the Jacobi iteration converge for this matrix? If so, what will its convergence factor be?

ii. Will the Gauss-Seidel iteration converge for this matrix?

iii. For which values of ω will the SOR iteration converge?

solution:

(a) For any fixed j, we have

$$\begin{aligned}\alpha\sin(kj\theta)-(\sin((k-1)j\theta)+sin((k+1)j\theta)) &= \alpha\sin(kj\theta)-2\cos(j\theta)\sin(kj\theta)\\ &= [\alpha-2\cos(j\theta)]\sin(kj\theta)\end{aligned}$$

Here, $\sin(0j\theta)=\sin((n+1)j\theta)=0$.
T_α is positive definite iff all $\lambda_j>0$, i.e., $\alpha>\max\limits_{j=1,\dots,n}\{2\cos(j\theta)\}=2\cos(\frac{\pi}{n+1})$

(b) i. Using Jacobi iteration, the iteration matrix is

$$G^{(Jac)}=\begin{pmatrix} 0 & -\frac{1}{2} & & & \\ -\frac{1}{2} & 0 & -\frac{1}{2} & & \\ & -\frac{1}{2} & \ddots & \ddots & \\ & & \ddots & \ddots & -\frac{1}{2} \\ & & & -\frac{1}{2} & 0 \end{pmatrix},$$

Similarly, we can show that the eigenvalues of $G^{(Jac)}$ are

$$\mu_j=-\cos(j\theta),\quad j=1,2,\dots,n$$

and $\rho(G^{(Jac)})=\cos(\frac{\pi}{n+1})<1$

ii. Let $A=L+D+U$, apply Householder-John Theorem, $N=L+D, P=U$ and by (a), we know A and $N^*+N-A=D$ are both symmetric positive definite, thus the Gauss-Seidel iteration converges.

iii. $N=L+\frac{1}{\omega}D, P=\frac{1}{\omega}D-(D+U)$, this time $N+N^*-A=(\frac{2}{\omega}-1)D$. Remember that in the class, we have proved that the necessary condition of the convergence of SOR method is $0<\omega<2$. By Householder-John theorem, this condition is also sufficient.

5. Let B be a matrix with the following structure:

$$B=\begin{pmatrix} O & B_{12} \\ B_{21} & O \end{pmatrix},$$

and let L and U be the lower and upper triangular parts of B, respectively.

(a) If μ is an eigenvalue of B, then so is $-\mu$.

(b) The eigenvalues of the matrix

$$B(\alpha)=\alpha L+\frac{1}{\alpha}U$$

defined for $\alpha\neq 0$ are independent of α.

solution:

(a) Suppose $\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$ is an eigenvector associated with μ, then $\begin{pmatrix} \vec{u} \\ -\vec{v} \end{pmatrix}$ is an eigenvector associated with $-\mu$.

(b) For any $\alpha \neq 0$, $B(\alpha)$ is similar to B, i.e., $B(\alpha) = XBX^{-1}$ with X defined by

$$\begin{pmatrix} 1 & O \\ O & \alpha \end{pmatrix}$$

This proves the desired result.