Lecture 8

Example: Consider: $\overset{\in\mathbb{R}}{a}\,\dfrac{d^2u}{dx^2} + \overset{\in\mathbb{R}}{b}\,\dfrac{du}{dx} = f$ for $x \in [0, 2\pi]$.

This time, we approximate $\dfrac{d^2u}{dx^2}$ by:

$$(*) \quad \frac{d^2u}{dx^2}(x_j) \approx \frac{u_{j-2} - 2u_j + u_{j+2}}{4h^2} \quad \text{for } j = 0, 1, 2, \dots, N-1$$

Again, we assume $u_{-1} = u_{N-1}$, $u_1 = u_{N+1}$, $u_{-2} = u_{N-2}$, ..., etc

Motivation: ① $u(x_j + 2h) \approx u(x_j) + 2h\,u'(x_j) + 2h^2 u''(x_j)$

② $u(x_j - 2h) \approx u(x_j) - 2h\,u'(x_j) + 2h^2 u''(x_j)$

① + ② : $u(x_{j+2}) + u(x_{j-2}) - 2u(x_j) = 4h^2\, u''(x_j)$

This time, we approximate $\dfrac{du}{dx}$ as:

$$(**) \quad \frac{du}{dx}(x_j) \approx \frac{u_{j+1} - u_{j-1}}{2h}$$

Denote (**) in matrix form as:

$$\begin{pmatrix} u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{N-1}) \end{pmatrix} = \tilde{D}\vec{u} \quad \text{where} \quad \tilde{D} = \frac{1}{2h}\begin{pmatrix} 0 & 1 & & & -1 \\ -1 & 0 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{pmatrix}$$

Denote (*) in matrix form as:

$$\begin{pmatrix} u''(x_0) \\ u''(x_1) \\ \vdots \\ u''(x_{N-1}) \end{pmatrix} = D\vec{u} \quad \text{where} \quad D = \frac{1}{4h^2}\begin{pmatrix} -2 & 0 & 1 & \cdots & 1 & 0 \\ 0 & -2 & 0 & 1 & & 1 \\ 1 & 0 & \ddots & 0 & \ddots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & \ddots & 1 \\ & & & & & 0 \\ 0 & 1 & & 1 & 0 & -2 \end{pmatrix}$$

Remark: $D = \tilde{D}^2.$

Last time:

<u>Claim:</u> $\vec{e^{ikx}}$ is an eigenvector of $\tilde{D}$ and D.

More specifically,

$$\left(D\vec{e^{ikx}}\right)_j = \underbrace{\left(-\frac{\sin^2(kh)}{h^2}\right)}_{\tilde{\lambda}_k^2} e^{ikx_j} \qquad \therefore\ D\vec{e^{ikx}} = \tilde{\lambda}_k^2 \vec{e^{ikx}}.$$

$$\left(\tilde{D}\vec{e^{ikx}}\right)_j = \underbrace{\left(\frac{i\sin(kh)}{h}\right)}_{\tilde{\lambda}_k} e^{ikx_j} \qquad \therefore\ \tilde{D}\vec{e^{ikx}} = \tilde{\lambda}_k \vec{e^{ikx}}$$

Now, $a \frac{d^2u}{dx^2} + b \frac{du}{dx} = f$ can be discretized as:

$$a \tilde{D}^2 \vec{u} + b \tilde{D} \vec{u} = \vec{f}$$ subject to the periodic condition.

Recall: $\{\overrightarrow{e^{ikx}}\}_{k=0}^{N-1}$ is a basis for $\mathbb{C}^N$

Again, let $\vec{u} = \sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\vec{f} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$

Then: $$a \tilde{D}^2 \left(\sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \right) + b \tilde{D} \left(\sum_{k=0}^{N-1} \hat{u}_k \overrightarrow{e^{ikx}} \right) = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

$$\Longleftrightarrow \sum_{k=0}^{N-1} \left(a \tilde{\lambda}_k^2 + b \tilde{\lambda}_k \right) \hat{u}_k \overrightarrow{e^{ikx}} = \sum_{k=0}^{N-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

Comparing coefficients, we get: $\left(a \tilde{\lambda}_k^2 + b \tilde{\lambda}_k \right) \hat{u}_k = \hat{f}_k$ for $k = 0, \ldots, N-1$

(algebraic equation)

Remark: $$a\tilde{\lambda}_k^2 + b\tilde{\lambda}_k = a\left(-\frac{\sin^2 kh}{h^2}\right) + b\left(i\,\frac{\sin kh}{h}\right)$$

$\therefore$ $a\tilde{\lambda}_k^2 + b\tilde{\lambda}_k = 0$ if and only if $\sin kh = 0$

$\because$ $h = \frac{2\pi}{N}$

$\therefore$ $\sin kh = 0 \iff \sin\left(k\frac{2\pi}{N}\right) = 0 \iff k = 0$ or $k = \frac{N}{2}$

(for $k = 0, 1, 2, \ldots, N-1$)

(assuming N is even)

For $k=0$ and $\frac{N}{2}$, $\tilde{\lambda}_k = 0$. We set $\hat{u}_0 = 0 = \hat{u}_{\frac{N}{2}}$.

In general, we set:

$$\hat{u}_k = \begin{cases} \hat{f}_k / (a\tilde{\lambda}_k^2 + b\tilde{\lambda}_k) & \text{for } k \neq 0, N/2 \\ 0 & \text{for } k=0 \text{ or } k = N/2 \end{cases}$$

$\vec{u}$ can be obtained by inverse DFT:

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = A_\omega \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{N-1} \end{pmatrix}$$

<u>Question:</u> How about the general solution?

<u>Answer:</u> Examine $N(a\tilde{D}^2 + b\tilde{D})$, the null space.

<u>Claim</u>: $N(a\tilde{D}^2 + b\tilde{D}) = \text{span}\{\overrightarrow{e^{i(0)x}}, \overrightarrow{e^{i(\frac{N}{2})x}}\}$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} e^{i(\frac{N}{2})x_1} \\ \vdots \\ e^{i(\frac{N}{2})x_{N-1}} \end{pmatrix} \right\}$$

<u>Proof</u>: $a\tilde{D}^2 + b\tilde{D}$ has <u>two</u> eigenvectors whose eigenvalue is 0.

These eigenvectors are $\overrightarrow{e^{i(0)x}}$ and $\overrightarrow{e^{i(\frac{N}{2})x}}$.

$N(a\tilde{D}^2 + b\tilde{D})$ = eigenspace of eigenvalue = 0

$$= \text{span}\{\overrightarrow{e^{i(0)x}}, \overrightarrow{e^{i(\frac{N}{2})x}}\}.$$

Thus, general sol: $\vec{u}^* = \vec{u} + c_1 \overrightarrow{e^{i(0)x}} + c_2 \overrightarrow{e^{i(\frac{N}{2})x}}$

for some c_1 and c_2.

c_1 and c_2 can be determined by certain conditions (such as boundary conditions)

Main idea of numerical spectral method

Diff. eqt $\longrightarrow$ Algebraic eqt $\longrightarrow$ solution $\vec{u}$

(e.g. $\frac{d^2u}{dx^2} = f$)

DFT on $\vec{f}$ $\left(\frac{\overline{A_w}}{n}\vec{f}\right)$ Fast?

Fast!

inverse DFT on $\hat{u}$ $\left(A_w \hat{u}\right)$ Fast?

Remark: • To develop an efficient numerical spectral method, we need to compute $A_w \hat{u}$ and $\frac{\overline{A_w}}{n}\vec{f}$ fast.

• Computational cost for $\underset{(n\times n)}{A_w}\hat{u}$ is $O(n^2)$.

Goal: Reduce the computational cost to $O(n\log n)$

e.g. $n = 2^{10}$, $n^2 = 2^{20}$, $n\log n = 10\cdot 2^{10} < 2^{14}$. $\therefore$ $2^6 = 64$ times faster!