Lecture 23:

Proof of lemma

Lemma: In the conjugate gradient algorithm:

$$W_k := \text{Span}\{\vec{p}_0, \vec{p}_1, \ldots, \vec{p}_{k-1}\} = \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-1}\} = \text{Span}\{\vec{r}_0, A\vec{r}_0, \ldots, A^{k-1}\vec{r}_0\}$$

Proof: Note that $W_k := \text{span}\{\vec{p}_0, \vec{p}_1, \ldots, \vec{p}_{k-1}\}$ where

$$\vec{p}_j = -\vec{r}_j - \beta_j \vec{p}_{j-1}$$

$\therefore$ $\vec{r}_j \in \text{Span}\{\vec{p}_j, \vec{p}_{j-1}\}$ for all $j = 1, 2, \ldots$

$-p_0$ $\quad (b'_0\vec{p}_0 + b'_1\vec{p}_1)$

Thus, for any $\vec{x} \in \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-1}\}$, $\vec{x} = a_0\vec{r}_0 + a_1\vec{r}_1 + \ldots + a_{k-1}\vec{r}_{k-1}$

$(b_0^{k-1}\vec{p}_{k-2} + b_1^{k-1}\vec{p}_{k-1})$

$\therefore$ $\vec{x} \in \text{Span}\{\vec{p}_0, \ldots, \vec{p}_{k-1}\}$.

We conclude that $\text{Span}\{\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_{k-1}\} \subseteq \text{Span}\{\vec{p}_0, \vec{p}_1, \ldots, \vec{p}_{k-1}\}$

Conversely, we will show that

$$\mathrm{Span}\{\vec{p}_0, \vec{p}_1, \dots, \vec{p}_{k-1}\} \subseteq \mathrm{Span}\{\vec{r}_0, \vec{r}_1, \dots, \vec{r}_{k-1}\}.$$

We will prove by mathematical induction on k.

When $k=1$, $\vec{p}_0 = -\vec{r}_0$. $\therefore$ $\mathrm{Span}\{\vec{p}_0\} = \mathrm{Span}\{\vec{r}_0\}$.

The statement is true when $k=1$.

Assume the statement is true for $k-1$. (induction hypothesis)

(That is : $W_{k-1} := \mathrm{Span}\{\vec{p}_0, \dots, \vec{p}_{k-2}\} \subseteq \mathrm{Span}\{\vec{r}_0, \dots, \vec{r}_{k-2}\}$)

For k,

$$\mathrm{Span}\{\vec{p}_0, \dots, \vec{p}_{k-2}, \vec{p}_{k-1}\} \subseteq \mathrm{Span}\{\vec{r}_0, \dots, \vec{r}_{k-2}, \vec{p}_{k-1}\}.$$

Observe that $\vec{p}_{k-1} = -\vec{r}_{k-1} - \beta_{k-1}\vec{p}_{k-2}$ and $\vec{p}_{k-2} \in \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-2}\}$

$$\therefore \quad \vec{p}_{k-1} \in \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-2}, \vec{r}_{k-1}\}$$

Thus,

$$\begin{aligned}\text{Span}\{\vec{p}_0, \ldots, \vec{p}_{k-2}, \vec{p}_{k-1}\} &\subseteq \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-2}, \vec{p}_{k-1}\}\\ &\subseteq \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-2}, \vec{r}_{k-1}\}\end{aligned}$$

By M.I., the statement is true for all k.

We can now conclude that:

$$\text{Span}\{\vec{p}_0, \ldots, \vec{p}_{k-1}\} = \text{Span}\{\vec{r}_0, \ldots, \vec{r}_{k-1}\}$$

Now, we will show that:

$$W_k = \text{Span}\{\vec{r}_0, A\vec{r}_0, \dots, A^{k-1}\vec{r}_0\}$$

The statement is obviously true for $k=1$.

Suppose the statement is true for $k-1$.

That is, $W_{k-1} = \text{Span}\{\vec{p}_0, \dots, \vec{p}_{k-2}\} = \text{Span}\{\vec{r}_0, \dots, \vec{r}_{k-2}\}$

$= \text{Span}\{\vec{r}_0, \dots, A^{k-2}\vec{r}_0\}$

For k, recall that $\vec{r}_{k-1} = \vec{r}_{k-2} + \alpha_{k-2} A\vec{p}_{k-2}$ $(*)$

By induction hypothesis:

$$\text{Span}\{\vec{r}_0, \vec{r}_1, \dots, \vec{r}_{k-2}, \vec{r}_{k-1}\} \subseteq \text{Span}\{\vec{r}_0, \dots, \vec{r}_{k-2}, A\vec{p}_{k-2}\}$$

Now, $\vec{p}_{k-2} \in \text{Span}\{\vec{r}_0, \dots, A^{k-2}\vec{r}_0\}$ (induction hypothesis)

$\therefore \; A\vec{p}_{k-2} \in \text{Span}\{A\vec{r}_0, \dots, A^{k-1}\vec{r}_0\}$.

Thus,

$$\text{Span}\{\vec{r}_0, \vec{r}_1, \dots, \vec{r}_{k-2}, \vec{r}_{k-1}\} \subseteq \text{Span}\{\overbrace{\vec{r}_0, \dots, \vec{r}_{k-2}}^{\text{Span}\{\vec{r}_0, \dots, A^{k-2}\vec{r}_0\} \;\subseteq}, \underbrace{A\vec{p}_{k-2}}_{\in \;\text{Span}\{A\vec{r}_0, \dots, A^{k-1}\vec{r}_0\}}\}$$

$$\subseteq \text{Span}\{\vec{r}_0, A\vec{r}_0, \dots, A^{k-1}\vec{r}_0\}$$

Also, by induction hypothesis,

$$A^{k-2}\vec{r}_0 \in \mathrm{Span}\{\vec{p}_0, \dots, \vec{p}_{k-2}\}$$

$$\therefore \quad A^{k-1}\vec{r}_0 \in \mathrm{Span}\{\underset{\overset{\cap}{\mathrm{Span}\{\vec{r}_0,\vec{r}_1\}}}{A\vec{p}_0}, \dots, \underset{\overset{\cap}{\mathrm{Span}\{\vec{r}_{k-2},\vec{r}_{k-1}\}}}{A\vec{p}_{k-2}}\}$$

$$\therefore \quad \mathrm{Span}\{\underbrace{\vec{r}_0, \dots, A^{k-2}\vec{r}_0}_{\overset{\cap\!|}{\mathrm{Span}\{\vec{r}_0, \dots, \vec{r}_{k-2}\}}}, \underbrace{A^{k-1}\vec{r}_0}_{\overset{\cap}{\mathrm{Span}\{\vec{r}_{k-2},\vec{r}_{k-1}\}}}\}$$

$$\subseteq \mathrm{Span}\{\vec{r}_0, \dots, \vec{r}_{k-1}\}$$

$$\therefore \quad \mathrm{Span}\{\vec{r}_0, \dots, \vec{r}_{k-1}\} = \mathrm{Span}\{\vec{r}_0, \dots, A^{k-2}\vec{r}_0, A^{k-1}\vec{r}_0\}$$

By M.I., the lemma is true!!