# Lecture 16:

**Recall:** Assume $A$ is diagonalizable. We can estimate the spectral radius $\rho(A)$ of $A$ as follows:

Start with an initial vector $\vec{x}^{(0)}$.

Consider the iterative scheme: $\vec{x}^{(k+1)} = \dfrac{A\vec{x}^{(k)}}{\|A\vec{x}^{(k)}\|_\infty}$ for $k = 0, 1, \ldots$

$$\rho_k = \|A\vec{x}^{(k)}\|_\infty \to \rho(A) \text{ as } k \to \infty$$

(assuming that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$)

**Example:** Consider: $A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & 10 \end{pmatrix}$. Using power method with $\vec{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, find the spectral radius of $A$.

**Solution:** $\vec{x}^{(1)} = \dfrac{A\vec{x}^{(0)}}{\|A\vec{x}^{(0)}\|_\infty} = \dfrac{(6, 8, 12)^T}{\|(6, 8, 12)^T\|_\infty} = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{pmatrix}$.

$\vec{x}^{(2)} = \dfrac{A\vec{x}^{(1)}}{\|A\vec{x}^{(1)}\|_\infty} = \dfrac{(7/3, 10/3, 16/3)^T}{\|(7/3, 10/3, 16/3)^T\|_\infty} = \begin{pmatrix} 7/16 \\ 5/8 \\ 1 \end{pmatrix}$.

Compute $\|A\vec{x}^{(k)}\|_\infty$, we have:

| $k$ | 6 | 8 | 10 |
|---|---|---|---|
| $\|A\vec{x}^{(k)}\|_\infty$ | 4.025316 | 4.006270 | 4.001564 |

. The dominant eigenvalue $\approx 4$

## How about if $A$ is NOT diagonalizable?

(Jordan Canonical Form)

Let $A = VJV^{-1}$, $J$ = Jordan Canonical Form

Since the dominant eigenvalue, $\lambda_1$, has multiplicity 1, the first Jordan block of $A$ must be a $1 \times 1$ matrix.

$$\therefore \quad J = \begin{pmatrix} \lambda_1 & & & \\ & [J(\lambda_{i_2})] & & \\ & & \ddots & \\ & & & [J(\lambda_{i_s})] \end{pmatrix}$$

Recall:

$$J(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Assuming that $\vec{x}^{(0)} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$, where $\{\vec{v}_1, \dots, \vec{v}_n\}$ are column vectors of $V$, which are linearly independent.

Note: $\vec{v}_1$ = eigenvector of $\lambda_1$

$$\left( A \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}}_{V} = \begin{pmatrix} | & | & \\ \vec{v}_1 & \vec{v}_2 & \cdots \\ | & | & \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & & \\ \vdots & & & \end{pmatrix}}_{J} \xRightarrow{\text{Compare } 1^{st} \text{ col}} A\vec{v}_1 = \lambda_1 \vec{v}_1 \right)$$

Assume $c_1 \neq 0$, then:

$$\vec{x}^{(k)} = \frac{A^k \vec{x}^{(0)}}{\|A^k \vec{x}^{(0)}\|_\infty} = \frac{(VJV^{-1})^k \vec{x}^{(0)}}{\|(VJV^{-1})^k \vec{x}^{(0)}\|_\infty}$$

$$= \frac{(VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)}{\|(VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)\|_\infty}$$

$$= \frac{VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n)}{\|VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\|_\infty} \quad \left( \because V^{-1}\vec{v}_j = \vec{e}_j = \begin{array}{r} \\ \\ \\ j^{th} \rightarrow \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

$$= \frac{c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right)}{\|c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right)\|_\infty}$$

(Note: $J^k\vec{e}_1 = \lambda_1^k\vec{e}_1$
$\Rightarrow VJ^k\vec{e}_1 = \lambda_1^k\vec{v}_1$ )

Now, $$\left(\frac{1}{\lambda_1} J\right)^k = \begin{pmatrix} 1 & & & \\ & \left[\frac{1}{\lambda_1} J(\lambda_{i_2})\right]^k & & \\ & & \ddots & \\ & & & \left[\frac{1}{\lambda_1} J(\lambda_{i_s})\right]^k \end{pmatrix} \to \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

as $k \to \infty$

$$\left|\frac{\lambda_{i_2}}{\lambda_1}\right| < 1 \qquad \left|\frac{\lambda_{i_k}}{\lambda_1}\right| < 1$$

$$\therefore \quad \vec{x}^{(k)} \approx \frac{c_1 \lambda_1^k \vec{v}_1}{\|c_1 \lambda_1^k \vec{v}_1\|_\infty}$$ when $k$ is large.

$$\therefore \quad \|A \vec{x}^{(k)}\|_\infty \to |\lambda_1|$$ as $k \to \infty$ as before!

# Generalization of Power method

Consider an invertible matrix $A$. Suppose $A$ has eigenvalues :

$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| \; (>0)$

Consider $A^{-1}$ (exist as all eigenvalues are non-zero). Then $A^{-1}$ has eigenvalues:

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ with $\left|\frac{1}{\lambda_n}\right| > \left|\frac{1}{\lambda_{n-1}}\right| > \cdots > \left|\frac{1}{\lambda_1}\right|$

Extension: Apply Power's method on $A^{-1}$ to obtain $\left|\frac{1}{\lambda_n}\right|$.

$\therefore$ the minimal eigenvalue can be determined! (Inverse Power method)

Remark: Computing $A^{-1}$ is difficult! We solve: $A\vec{y} = \vec{x}^{(n)}$ in each iteration to determine $A^{-1}\vec{x}^{(n)}$.

Finding $A^{-1}$ is equivalent to solving:

$$A\vec{y} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad A\vec{y} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

$$A \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}}_{A^{-1}} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$