MATH3290 2024/25
Mathematical Modeling

Chapter 1

Difference equation model $\Delta a_n = a_{n+1} - a_n = f(a_n)$.
The least-squares method for parameter estimation.
Examples: Biomass growth, drug decay.
Equilibrium value (EV): find a such that $f(a) = a$.
For $a_{n+1} = ra_n + b$ ($r \neq 1$), the EV $a = \frac{b}{1-r}$.
Systems of difference equations, equilibrium values.
Examples: rental car company.

Chapter 3

Model fitting (curve may not meet the points) vs interpolation (curve goes through all points).
Chebeshev criterion: minimize max error.
Least-squares criterion: minimize sum of squared errors.
Applying least-squares: The error function

$$S(p_1, \cdots, p_k) = \sum_{i=1}^{m} (y_i - f(x_i; p_1, \cdots, p_k))^2$$

To mimize S:

$$\frac{\partial S}{\partial p_j} = -2 \sum_{i=1}^{m} (y_i - f(x_i; p_1, \cdots, p_k)) \frac{\partial f}{\partial p_j} = 0$$

Examples: $f(x; a, b) = ax + b$, $f(x; a, b) = ag(x) + bh(x)$, $f(x; a, b) = be^{ax}$.

Chapter 4

One-term models: $g(y) = af(x) + b$. Find a, b using the least-squares.
High-order polynomial models: use Lagrangian form

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

where the Lagrangian basis

$$L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

satisfies $L_k(x_k) = 1$ and $L_k(x_j) = 0$ for $j \neq k$.
Low-order polynomial models by constructing difference tables.
Cubic splines: piecewise cubic polynomials connecting data points $\{(x_i, y_i)\}_{i=0}^n$, ensuring continuous first and second derivatives.
Natural boundary condition: second derivatives at endpoints are zero: $S''(x_0) = S''(x_n) = 0$.

Chapter 7

Linear programming problem (standard form):

$$\begin{aligned} \max/\min \quad & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} g_{ji} x_i \leq b_j, \quad j = 1, \cdots, m \end{aligned}$$

where $x_1, \cdots, x_n \geq 0$.
Simplex method: initialization, optimality test, feasibility test, pivot.

Chapter 13

Unconstrained optimization:

$$\max/\min \quad f(x_1, \cdots, x_n)$$

Optimality conditions: $f_{x_1} = \cdots = f_{x_n} = 0$.
Use Hessian matrix to determine local max/min.
Constrained optimization:

$$\begin{aligned} \max \quad & f(x_1, \cdots, x_n) \\ \text{s.t.} \quad & g(x_1, \cdots, x_n) = 0, \quad h(x_1, \cdots, x_n) \geq 0 \end{aligned}$$

Use Lagrange multiplier $L = f + \lambda g + \mu h$. Optimality conditions:

$$L_{x_1} = \cdots = L_{x_n} = 0, \quad L_\lambda = g = 0$$

plus the KKT conditions:

$$L_\mu = h \geq 0, \quad \mu \geq 0, \quad \mu L_\mu = 0.$$

Minimization problem and inequality constraints in the form of $h(x_1, \cdots, x_n) \leq 0$: conver it to the above form.
Sensitivity analysis: how the optimal solution changes when the parameters change.
Gradient method:

$$x_{k+1} = x_k \pm \lambda_k f_x(x_k, y_k), \quad y_{k+1} = y_k \pm \lambda_k f_y(x_k, y_k)$$

Chapter 8

Graph $G = (V, E)$. Directed and undirected edges.
Shortest path problem: Dijkstra's algorithm, dynamic programming.
Example: equipment replacement cost.
Maximum flow problem: Ford and Fulkerson algorithm.
Examples: assignment problem, matrix 0-1 problem.

Chapter 11

Differential equation model $\frac{dy}{dx} = g(x, y)$.
Solve the model via **separation of variables**.
Examples: population models $\frac{dP}{dt} = kP$ and $\frac{dP}{dt} = rP(M - P)$.
Equilibrium point (EP) of $\frac{dy}{dx} = f(y)$: $f(y^*) = 0$.
Phase lines: analyze signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
Stability of EP via phase lines.
Euler's method for numerical solutions:

$$y_{n+1} = y_n + g(x_n, y_n)\Delta x$$

Parameter estimation using the least-squares method.

Chapter 12

System of differential equations:

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

Equilibrium point (EP): $f(x^*, y^*) = g(x^*, y^*) = 0$.
Graphical analysis in 2D: analyze signs of $\frac{dx}{dt}, \frac{dy}{dt}$.
Stability of EP: linearization and analysis on eigenvalues of the Jacobian matrix.
Euler's method for numerical solutions.
Examples: competitive hunter model, predator-prey model.