MATH 3290 Mathematical Modeling

Chapter 2: The Modeling Process

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Introduction

Why do we need mathematical models?

Assume that you need to find the result of some real-world problems.

Two possibilities:

- conduct real-world experiments and obtain the results,
- obtain the results by using mathematical modeling.

Comparison of two possibilities

- Conduct real-world experiments and obtain the results.
  - 👍 You get the "exact" results (e.g. drug concentration).
  - 👎 Too expensive to perform an experiment (e.g. to build an airplane).
  - 👎 It is impossible to perform an experiment (e.g. to predict the future weather).
- Obtain the results by using mathematical modeling.
  - 👍 Results are obtained easily (through the use of mathematical knowledge),
  - 👎 Due to simplification, the results may be inaccurate (usually a sufficiently refined model will give good results).

Construction of models

Six major steps in mathematical modeling.

Step 1 identify the problem

- What is the mathematical nature of the problem?

Step 2 make assumptions

- We cannot take all factors into account (the model will be too complicated, sometimes over-fitted);
- identify the independent (input) and dependent (output) variables;
- neglect some variables that give smaller influence;
- find a mathematical relationship among these variables.

Step 3 solve the model

- Find the solution of the mathematical problem (our focus).

Step 4 verify the model

- Test the results (dependent variables) of the model with real data.
- Note that a model is not a universal law, it is applicable under the assumptions made (in **Step 2**).

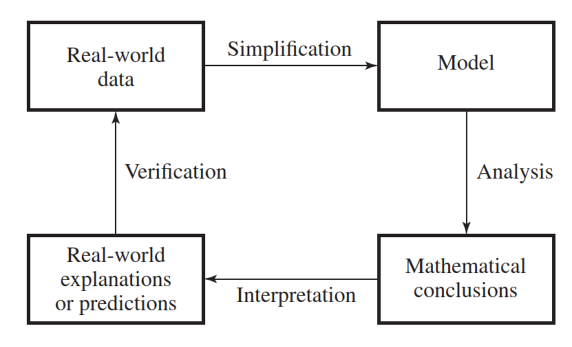
Step 5 implement the model

- Write a computer code, and use it (also our focus).

Step 6 maintain the model

- When necessary, revise the assumptions (e.g. interest rate may change in a finance model).

Modeling process



The area of mathematical modeling is connected to mathematics, science, and engineering.

Kepler's laws of planetary motion

First law The orbit of every planet is an ellipse with the Sun at one of the two focuses.

Second law A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Third law The ratio of the square of an object's orbital period with the cube of the semi-major axis of its orbit is the same for all objects orbiting the same primary.

Johannes Kepler (1571-1630)

Kepler's laws of planetary motion

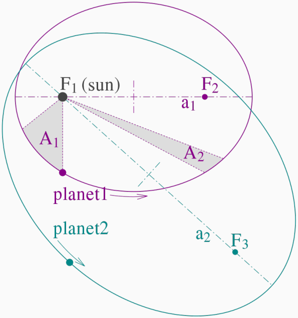


Illustration of Kepler's laws.

Johannes Kepler (1571-1630)

The third law

Kepler published his first two laws in 1609. The third law didn't come along until ten years later, in 1619.

Step 1 : identify the problem

Kepler wanted to find a relation of planet's orbital period and distance to the Sun.

In Kepler's time, only those data are available. Considering more additional factors (such as the mass) is impossible.

The third law

Step 2 : make assumptions

The orbital period T should be proportional to some powers of the distance r, that is

$$T \propto r^{\alpha}.$$

In Kepler's time, mathematics are too elementary to play some advanced tricks, even logarithm is just invented.

The third law

Step 3 : solve the model

Kepler had the research records from Tycho Brahe who was his former boss.

Data used by Kepler (1618)

Planet	r (AU)	T (days)	r^3/T^2 (10^{-6}AU^3/day^2)
Mercury	0.389	87.77	7.64
Venus	0.724	224.70	7.52
Earth	1.00	365.25	7.50
Mars	1.524	686.95	7.50
Jupiter	5.20	4332.62	7.49
Saturn	9.510	10759.2	7.43

Tycho Brahe (1546-1601)

Recall that logarithm was just invented!

The third law

"On the 8th of March of this year 1618, if exact information about the time is desired, it appeared in my head. But I was unlucky when I inserted it into the calculation, and rejected it as false. Finally, on May 15, it came again and with a new onset conquered the darkness of my mind, whereat there followed such an excellent agreement between my seventeen years of work at the Tychonic observations and my present deliberation that I at first believed that I had dreamed and assumed the sought for in the supporting proofs. But it is entirely certain and exact that the proportion between the periodic times of any two planets is precisely one and a half times the proportion of the mean distances."
—Max Caspar, Kepler, 1993

The third law

Step 4: verify the model

In 1621, Kepler noted that his third law applies to the four brightest moons of Jupiter.

For **Step 5** and **Step 6**, we now have Newton's law of universal gravitation (also applying the formula of centrifugal force):

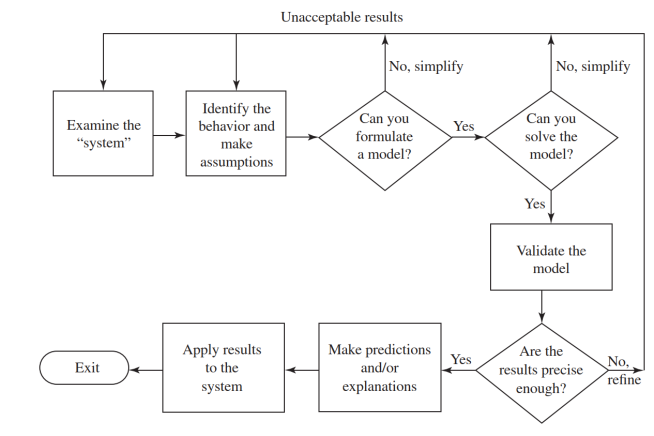
$$F = G\frac{m_s m_p}{r^2} = m_p \omega^2 r \Rightarrow \omega = \frac{\sqrt{G m_s}}{r^{3/2}},$$

$$T = \frac{2\pi}{\omega} \Rightarrow T = \frac{2\pi}{\sqrt{G m_s}} r^{3/2},$$

where G is the gravitational constant and ω is the angular velocity.

Iterative nature

Model construction is an iterative process.



Simplification vs refinement

How to simplify or refine the model?

Model simplification	Model refinement
1. Restrict problem identification.	1. Expand the problem.
2. Neglect variables.	2. Consider additional variables.
3. Conglomerate effects of several variables.	3. Consider each variable in detail.
4. Set some variables to be constant.	4. Allow variation in the variables.
5. Assume simple (linear) relationships.	5. Consider nonlinear relationships.
6. Incorporate more assumptions.	6. Reduce the number of assumptions.

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